Analytic Models of Near-Field RF Sheaths

D. A. D’Ippolito and J. R. Myra
Lodestar Research Corporation

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Background

- We need quantitative calculations of rf antenna sheaths with full plasma effects; sheath BC is a promising approach.

- Here we describe an analytic model of antenna sheaths using this new BC (useful for code benchmarking); two cases:
  - tenuous plasma limit (D’Ippolito & Myra, PoP 022506, 2009)
  - full plasma dielectric

- Quantitative results of the model:
  - sheath capacitance effects, important when \( \Lambda \equiv \varepsilon_{||} (\Delta / L) \gg 1 \)
  - limits of validity of the vacuum field approx. \( V_{\text{vac}} = -\int dz \, E_{||}^{(\text{vac})} \)
  - screening of \( E_{||} \) by sheaths when \( \varepsilon_{||} \gg 1 \)
Approximations

Simple model of ICRF wave launcher:

- **electromagnetic wave propagation** in a plasma-filled waveguide
- constant-density plasma (e.g. assume strong rf convection)
- FW is polarized in the y direction and propagates in the +x direction.
- equilibrium magnetic field is \( B = B_z e_z + B_y e_y \)
- **perturbation expansion** assuming weak field line tilt (\( b_y \equiv B_y / B \ll 1 \))
- Assume \( k_y = 0 \) \( \Rightarrow \) keep magnetic flux from current straps, but not effects of feeders or box currents (model valid near center of antenna at \( y = 0 \)).
- apply the sheath BC at \( z = \pm L \), where B field lines intersect the sheaths
- field line tilt, sheath BC \( \Rightarrow \) coupling of FW to SW
- Use low density (“tenuous plasma”) approximation (\( \epsilon_\perp = 1, \epsilon_x = 0 \) but \( |\epsilon_\parallel| \gg 1 \)) or full plasma dielectric (\( \epsilon_\perp \neq 1, \epsilon_x \neq 0 \), \( |\epsilon_\parallel| \gg 1 \))
Linear rf wave propagation

Wave operator:

\[ L = -\frac{c^2}{\omega^2} \nabla \times (\nabla \times E) + \varepsilon \cdot E = n \times (n \times E) + \varepsilon \cdot E = (nn - n^2I + \varepsilon) \cdot E \]

where \( n = k_c / \omega \)

and we expand the plasma dielectric in powers of \( b_y \):

\[ \varepsilon = \varepsilon_0 + \varepsilon_1, \quad \varepsilon_0 = I + (\varepsilon_{||} - \varepsilon_{\perp})e_z e_z, \quad \varepsilon_1 = (\varepsilon_{||} - \varepsilon_{\perp})(b_y e_z + e_z b_y) \]

Homogeneous plasma dispersion relation:

\[ (\varepsilon_{\perp} - n_{xf}^2 - n_{zf}^2)(\varepsilon_{\perp} - n_{zf}^2) - \varepsilon_x^2 = 0 \]

First, give the results for the tenuous plasma approximation and then give generalization to finite-density case.
RF field solution

- In lowest order, assume a propagating FW

\[ E_0 = E_f = \{0, \hat{E}_y \cos (k_{zf}z - \delta), 0\} e^{ik_x f x} \]

BC on \( E_y \) requires \( \cos (k_{zf}L - \delta) = 0 \) \( \Rightarrow \eta_{zf} \equiv k_{zf}L = \pi/2 + \delta \)

Parameter \( \delta \) determines the parity of \( E_y \) (\( \Rightarrow \) antenna phasing)
\( \delta = 0 \) \( \Rightarrow \) monopole phasing, \( \delta = \pi/2 \) \( \Rightarrow \) dipole phasing,

- In first order, the FW drives a wave with SW polarization but satisfying FW dispersion relation

\[ E_{1p} = C \{iG_f \sin (k_{zf}z - \delta), 0, \cos (k_{zf}z - \delta)\} e^{ik_x f x} \]

\[ G \equiv n_x n_z / (n_z^2 - 1) \]

\[ C = -b_y \hat{E}_y \]
Sheath BC

The sum of these waves \((E_{0f} + E_{1p})\) does NOT satisfy the full sheath BC. Need to add a **SW solution** \(E_{1s}\)

\[
E_{1s} = A \{iG_s \sin(k_{zs}z - \delta), 0, \cos(k_{zs}z - \delta)\} e^{ik_{xs}x}
\]

- **SW polarization**
- **SW wavenumbers** (coupling by sheath)

The **sheath BC** determines the amplitude \(A\) and requires \(k_{xs} = k_{xf}\):

\[
\left( E_x + \Delta \varepsilon_{||} \frac{\partial E_z}{\partial x} \right)_{z=L} = 0 \quad \Rightarrow \quad A = -C \frac{D_f}{D_s} = b_y \hat{E}_y \frac{G_f}{D_s} \propto \frac{E_{||}}{D_s}
\]

where \(D\) is the coupled sheath-plasma dispersion relation

\[
D(\eta_{zj}) \equiv G_j \sin(\eta_{zj} - \delta) + k_{xj} \Delta \varepsilon_{||} \cos(\eta_{zj} - \delta)
\]

- **plasma inductance**
- **sheath capacitance**

\[\Rightarrow\] **sheath-plasma resonance** when \(D \to 0\)
Computing sheath voltage

Commonly used “vacuum field sheath approximation”

\[ V_{sh} \equiv V_{vac} = -\int_0^L dz \, E_{||}^{(vac)} \]

gives the following sheath voltage (for monopole phasing)

\[ V_{vac} (\delta = 0) = -\int_0^L dz \, b_y \hat{E}_y \cos k_z f z \, e^{ik_x x} = -\frac{2}{\pi} b_y E_0 L \, e^{ik_x x} \]

We include plasma effects by using the SBC and integrating across the sheath

\[ V_{sh} \equiv -\int_{sh} dz \, E_{z}^{(sh)} = -\Delta \varepsilon_{||} E_{z}(L) \]  

(on plasma side of sheath-plasma boundary)

Ratio of the two model predictions is given by

\[ \hat{V}(\delta) \equiv \frac{V_{sh}(\delta)}{V_{vac}(0)} = \frac{(\Delta / L) n_z^2 \eta_0 \cos(\eta_0 - \delta)}{(\Delta / L)(n_z^2 - 1)\alpha \eta_0 \cos(\eta_0 - \delta) + \alpha \sin(\eta_0 - \delta)} \]

\[ \eta_0 \equiv \omega L / c \]

\[ \alpha \equiv 1 + (2\delta / \pi) \]
Screening of plasma $E_\parallel$ by sheaths

Continuity of $D_{normal}$ across sheath-plasma interface $\Rightarrow$ screening of $E_\parallel$

$$E_\parallel = \frac{V_{\text{vac}}(0)}{\Lambda \alpha L} \left( \frac{(\Delta / L)n_{zf}^2 \eta_0 \cos(k_{zsf}z - \delta)}{\sin(\eta_0 - \delta) + (\Delta / L)(n_{zf}^2 - 1)\eta_0 \cos(\eta_0 - \delta)} \right),$$

$$\Lambda \equiv -\varepsilon_\parallel \frac{\Delta}{L}$$

For monopole phasing with $n_{zf} >> 1$ and $\eta_0 = \omega L/c << 1$

$$E_\parallel(0) = \frac{V_{\text{vac}}(0)}{L} \frac{\Re}{\Lambda} \cos k_{zsf}z$$

where

$$\Re \equiv \frac{(\Delta / L)n_{zf}^2}{1 + (\Delta / L)n_{zf}^2} \leq 1$$

$\Rightarrow$ screening in strong sheath limit $(E_\parallel \to 0$ as $\Lambda \to \infty)$

voltage split

$$\frac{V_{\text{pl}}}{V_{\text{sh}}} \equiv \left| \frac{1}{L} \int_{z=-L} E_\parallel \right| / \int_{sh} E_\parallel = \frac{\tan \eta_0}{\Lambda \eta_{0s}} \to \frac{1}{\Lambda}$$

as $\eta_0 \to 0$
Self-consistent (Child-Langmuir) sheath solution

Sheath width must satisfy the Child-Langmuir (CL) Law

\[
\Delta = \lambda_D \left( \frac{eV_0}{T_e} \right)^{3/4}
\]

where “rectified” (dc) sheath voltage is \( V_0 = 3T_e + 0.6V_{sh} \)

Use nonlinear rootfinder to solve the following constraint (obtained by combining the CL Law with the expression for \( V_{sh} \) derived above)

\[
\frac{C \hat{V}^{1/4}}{A - B \hat{V}} = \left( \frac{\lambda_D}{L} \right) \left( \frac{eV_{vac}}{T_e} \right)^{3/4}
\]

where

\[
A = n_{zf}^2 \eta_0 \cos(\eta_0 - \delta)
\]

\[
B = (n_{zf}^2 - 1)\alpha \eta_0 \cos(\eta_0 - \delta)
\]

\[
C = \alpha \sin(\eta_0 - \delta)
\]
Numerical solution for self-consistent $V_{sh}$

Plot of the self-consistent (Child-Langmuir) sheath voltage, $V_{sh}/V_{vac}$, (obtained from solution of eq. on previous page) vs the “vacuum field” sheath drive, $eV_{vac}/T_e$, for monopole phasing with parameters $n_e = 10^{10}$ cm$^{-3}$, $T_e = 50$ eV, and $n_{zf} = 10$. 
Generalization to full plasma dielectric
(in monopole phasing)

General result: \[ V_{\text{sh}} = -\Delta \varepsilon || \left( -\hat{E}_y \frac{Q_f}{F_f} \sin \eta_{zf} + A \cos \eta_{zs} \right) \]

- \( b_y \neq 0 \) and \( \varepsilon_x = 0 \) (recover tenuous plasma result)
- \( b_y \neq 0 \) and \( \varepsilon_x \neq 0 \) (new finite-density effects)

From definition of \( V_{\text{sh}} \):
\[
\hat{V} = \left( \frac{n_{zf}^2}{n_{zf}^2 - \varepsilon_\perp} \right) \frac{n_x^2 (\Delta / L) \eta_{zs} \cos \eta_{zs}}{(1 - Q_f R_s) \sin \eta_{zs} - n_x^2 (\Delta / L) \eta_{zs} \cos \eta_{zs}}
\]

- Sheath BC + SW polarization
- From particular solution, driven by FW
- Sheath-plasma-wave physics

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Special cases (con’t)

\( b_y = 0, \ v_x \neq 0, \ \text{and} \ v_\perp \neq 1 \)  

(finite density terms enhance sheath drive; competitive when \( v_x > b_y L / \Delta \))

\[
V_{sh} = -\Delta n_x n_{zf} \hat{E}_y Q_f \sin \eta_f \left( 1 + \frac{k_x \Delta n_x n_{zs} \cos \eta_s}{(1 - Q_f R_s) \sin \eta_s - k_x \Delta n_x n_{zs} \cos \eta_s} \right)
\]

Here \( V_{sh} \propto Q_f k_x \Delta \hat{E}_y \propto v_x k_x \Delta \hat{E}_y \) is comparable to vacuum sheath voltage when

\[
\varepsilon_x \geq \frac{b_y L}{\Delta}
\]

Definitions:

\[
F_f = -\frac{\varepsilon_{||}}{n_x n_{zf}}, \quad F_s = -\frac{\varepsilon_{||}}{n_x n_{zs}}, \quad R_s \approx -\frac{\varepsilon_x}{n_x^2} = -Q_f
\]
Numerical results

- plot approximate result for $V_{sh}/V_{vac}$ with $b_y \neq 0$ and $\varepsilon_x \neq 0$, keeping only terms $\propto b_y$ in numerator
- CL nonlinear constraint NOT enforced here
- parameters: $k_z f L = \pi/2$ (lowest mode), $b_y = 0.1$, $B = 30$ kG, $T_e = 30$ eV, $L = 50$ cm, $f = 45$ MHz.
Summary

- We have derived an expression for the “antenna” sheath voltage (using a waveguide model) including:
  - sheath BC [D’Ippolito and Myra, PoP 2006, Myra et al., PoP 1994]
  - finite plasma effects (plasma dielectric)
  - Child-Langmuir physics (self-consistent sheath width)

- The model shows that:
  - the magnetic field tilt (not normal to FW vector) couples the FW & SW
  - sheath BC ⇒ FW-SW coupling is modified by the sheath capacitance when $\Lambda = \varepsilon_|| \Delta/L >> 1$.
  - the sheath capacitance also screens $E_||$ from the plasma when $\varepsilon_|| >> 1$.
  - finite density at the antenna has order unity effects on $V_{sh}$
  - “tenuous plasma” version is accurate for $n_e$ below the FW cut-off

- Applications of the model include:
  - calculating the self-consistent sheath voltage for benchmarking sheath BC in rf antenna codes
  - evaluating validity of the approximate “vacuum rf field” sheath model: valid when $(\Delta/L)n_{zf}^2 >> 1$; this appears to be hard to satisfy.
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http://www.lodestar.com/LRCreports

Recent papers on rf sheath modeling using the sheath BC:


• "Analytic model of near-field radio-frequency sheaths: II. Full Plasma Dielectric," D. A. D'Ippolito and J. R. Myra, manuscript in progress.
Appendix

Definitions of dielectric tensor elements
(and approximations)

\[ \varepsilon_\perp = 1 - \frac{\omega_{pi}^2}{\left(\omega^2 - \Omega_i^2\right)} \rightarrow 1 \quad (\text{tenuous plasma}) \]

\[ \varepsilon_x = \frac{\omega_{pi}^2\omega}{\Omega_i\left(\omega^2 - \Omega_i^2\right)} \ll 1 \quad (\text{tenuous plasma}) \]

\[ \varepsilon_\parallel = 1 - \frac{\omega_{pe}^2}{\omega^2} \gg n_x^2 \quad (\text{for all } n_e) \]