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Momentum Conservation and Nonlinear RF-Induced Flows

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Abstract. Recent progress on the numerical computation of 2D full-wave field solutions has motivated advances in the nonlinear theory of rf-induced plasma flows for the control of turbulence. Here, an accounting of how momentum is injected into the system by the antenna, and how it can be transferred from waves to plasma flows and to the equilibrium magnetic field coils and walls is given. Equations for both the plasma momentum and the wave momentum are developed. The former is a recapitulation of results from nonlinear flow drive theory. The latter equation yields a generalized form of the Maxwell stress tensor, including plasma dielectric effects. It is shown that momentum is conserved by the plasma-wave-antenna-wall system for poloidal and toroidal flux-surface-averaged flows. In general, however, momentum exchange with the equilibrium magnetic field coils is possible.

INTRODUCTION

Pioneering[1] as well as more recent[2-6] theoretical papers have considered the topic of rf-driven flows in tokamak plasmas. It has been suggested that ICRF waves could be employed both to flexibly control internal transport barriers, and also to enable fundamental physics investigations of nonlinear waves, flows and turbulence.

Recent advances in the numerical computation of rf fields have permitted full-wave solutions of ICRF fast waves undergoing mode conversion in 2D (axisymmetric) tokamak plasmas where the equilibrium poloidal magnetic field plays an important role.[5] In addition to the bipolar sheared-flow layers that were investigated previously, these solutions have illustrated the importance of direct wave-momentum absorption by the plasma leading to net (unipolar) flows. In this paper, we investigate the conservation laws for wave and plasma momentum, and address the question of when forces on the equilibrium magnetic field coils need be considered.

Waves of frequency ω cause plasma motion both on the rapid ω time scale (accounted for in the plasma dielectric) and on the slow ("dc") time scale, treated in the plasma momentum equation below. The external world interacts with the wave-plasma system through the Lorentz force $\mathbf{F} = \rho\mathbf{E} + (1/c)\,\mathbf{J} \times \mathbf{B}$. Each of the quantities, ρ, $\mathbf{E}$, $\mathbf{J}$ and $\mathbf{B}$, have both oscillatory (ω) and dc (slow) contributions. We account for the ω*ω products of external forces in the wave momentum equation and the dc*dc products in the plasma momentum equation in the sections which follow.

PLASMA MOMENTUM

The species-summed plasma momentum equation describes the evolution of plasma flows by the rf (plasma wave) force $\mathbf{F}_{pw}$,

$$\frac{\partial}{\partial t}(nm\mathbf{u}) + \nabla\cdot(nm\mathbf{uu}) + \nabla p + \nabla\cdot\overleftrightarrow{\Pi}_0 - \nabla\cdot\overleftrightarrow{D}\nabla(nm\mathbf{u}) = \frac{1}{c}\mathbf{J}\times\mathbf{B} + \mathbf{F}_{pw} \qquad (1)$$

Here $\mathbf{u}$ is the dc flow velocity of the plasma, $p = n(T_e + T_i)$ is the equilibrium plasma pressure, $\mathbf{J}$ is the dc plasma current, $\mathbf{B}$ is the equilibrium magnetic field, $\overleftrightarrow{\Pi}_0$ is a viscosity tensor (e.g. due to neoclassical physics) that describes the reaction of the plasma to the driven flows and $\overleftrightarrow{D}$ is a diffusion tensor (e.g. describing turbulent diffusion of toroidal momentum). In Eq. (1) we have neglected $\rho\mathbf{E} \sim \nabla^2\phi\nabla\phi$ relative to ∇p as it is smaller by $(\lambda_d/L)^2 << 1$ for a quasineutral plasma. Diffusion describes momentum loss of plasma flows to the wall.

The total nonlinear force of the waves on the plasma is[4,5]

$$\mathbf{F}_{pw} = \left\langle \rho_1\mathbf{E}_1 + \frac{1}{c}\mathbf{J}_1\times\mathbf{B}_1 - \nabla\cdot\overleftrightarrow{\Pi}_{ql}\right\rangle \equiv \mathbf{F}'_{pw} - \left\langle\nabla\cdot\overleftrightarrow{\Pi}_{ql}\right\rangle \qquad (2)$$

where $\rho_1 = \Sigma_j\, n_{1j}Z_je$ and $\mathbf{J}_1$ are the species summed charge density and current [$\sim\exp(-i\omega t)$, i.e. first order in wave fields], $\overleftrightarrow{\Pi}_{ql}$ is the nonlinear stress tensor[3] and <…> is a fast (ω) time average. Using only Maxwell's equations and defining $\mathbf{P} = \Sigma_j\, \overleftrightarrow{\chi}_j\cdot\mathbf{E}$ with dielectric susceptibility $\overleftrightarrow{\chi}$

$$\mathbf{F}_{pw} = \left\langle \frac{1}{16\pi}(\nabla\mathbf{E}^*)\cdot\mathbf{P} - \frac{1}{16\pi}\nabla\cdot(\mathbf{P}\,\mathbf{E} + \overleftrightarrow{\Pi}_{ql})\right\rangle \qquad (3)$$

After some algebra, in the case where parallel rf forces on ions are negligible,[4,5]

$$\mathbf{F}_{pw} = \mathbf{k}\frac{P_{rf}}{\omega} - \varepsilon^h\nabla U_0 - \nabla_\perp U_1 + \mathbf{b}\times\nabla U_2 \equiv \mathbf{F}_d - \nabla_\perp U_1 + \mathbf{b}\times\nabla U_2 \qquad (4)$$

where P_{rf} is the absorbed rf power density, $\mathbf{k}$ is the wavenumber (in general summed over all modes), ε^h is the Hermitian part of the dielectric tensor and U_0, U_1 and U_2 (whose exact forms are not needed here) are given as bilinear products of the wave electric field. The "direct" term $\mathbf{F}_d$, defined by Eq. (4), has a more general form in terms of sums over modes using the W tensor.[2,4] The U_1 and U_2 terms give rise to a redistribution of the plasma momentum by the waves (e.g. sheared flows with no net momentum input). The $\mathbf{k}P_{rf}/\omega$ term represents momentum exchange between the plasma and the waves. The U_0 term is the standard reactive ponderomotive potential term and does not give rise to flux-surface-averaged poloidal or toroidal flows because it has the form of a gradient in the parallel and toroidal directions. The flux-surface averaged poloidal and toroidal flows come from the direct ($\mathbf{k}P_{rf}/\omega$) and the dissipative stress (U_2) terms.

The terms on the lhs of Eq. (1) are in conservation law form, while those on the rhs contain momentum exchange with the coils and the waves. In general both waves and flows can induce plasma currents that cause the plasma to exchange momentum with the coils. This can be demonstrated by considering the case of flows guided through a turn by a curved magnetic field. Another example is the case of ponderomotive drift currents $\mathbf{J} \sim \mathbf{F}_{pw}\times\mathbf{B}$ induced by rf waves. However, the relevant toroidal and parallel flux-surface-averages of Eq. (1),[6] viz. $\langle\mathbf{B}\cdot(1)\rangle_\psi$ and $\langle R\mathbf{e}_\zeta\cdot(1)\rangle_\psi$ annihilate $\mathbf{J}\times\mathbf{B}$. Thus, momentum exchange between the plasma and the equilibrium magnetic field coils plays no role in understand *flux-surface-averaged* toroidal and poloidal flows.

FIELD AND WAVE MOMENTUM

The momentum conservation law for the electromagnetic fields is given by[7]

$$\frac{\partial}{\partial t}\left\langle \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\right\rangle + \nabla\cdot\left\langle \overleftrightarrow{T}_{em}\right\rangle = -\left\langle \rho\mathbf{E} + \frac{1}{c}\mathbf{J}\times\mathbf{B}\right\rangle \equiv -\mathbf{F}'_{pw} + \mathbf{F}_{ext} \qquad (5)$$

where the external force $\mathbf{F}_{ext}$ is defined in Eq. (10) and the Maxwell stress tensor is

$$\overleftrightarrow{T}_{em} = \frac{1}{4\pi}\left[\frac{1}{2}\overleftrightarrow{I}(E^2 + B^2) - (\mathbf{EE} + \mathbf{BB})\right] \qquad (6)$$

with $\overleftrightarrow{I}$ the identity tensor. When ρ and $\mathbf{J}$ consist of both plasma charges and currents (ρ_1 and $\mathbf{J}_1$) and "external" (i.e. antenna and wall, ρ_{ext} and $\mathbf{J}_{ext}$) charges and currents, then part of the plasma responses contained in $\mathbf{F}_{pw}'$ can be absorbed into the field momentum terms and Eq. (5) takes the form (after some algebra)

$$\frac{\partial}{\partial t}\left\langle \frac{\mathbf{D}\times\mathbf{B}}{4\pi c}\right\rangle + \nabla\cdot\left\langle \overleftrightarrow{T}_{w}\right\rangle + \left\langle \mathbf{F}'_d\right\rangle = \mathbf{F}_{ext} \qquad (7)$$

$$\overleftrightarrow{T}_{w} = \frac{1}{4\pi}\left[\frac{1}{2}\overleftrightarrow{I}(\mathbf{D}\cdot\mathbf{E} + B^2) - (\mathbf{DE} + \mathbf{BB})\right] \qquad (8)$$

$$\mathbf{F}'_d = -\frac{1}{8\pi}\left[(\nabla\mathbf{D})\cdot\mathbf{E} - (\nabla\mathbf{E})\cdot\mathbf{D}\right] \qquad (9)$$

$$\mathbf{F}_{ext} = -\left\langle \rho_{ext}\mathbf{E}_1 + \frac{1}{c}\mathbf{J}_{ext}\times\mathbf{B}_1\right\rangle \qquad (10)$$

where $\mathbf{D} = \mathbf{E} + \mathbf{P}$. Note that the nonlinear stress tensor $\overleftrightarrow{\Pi}_{ql}$ is not present in the wave momentum equation, while the vacuum part of the Maxwell stress tensor does not appear in the plasma momentum equation.

The terms in $\mathbf{F}_d'$ are not manifestly conservative, and contain the physics of dc momentum exchange between the waves and the plasma. For example, in the eikonal limit, to zero order in the wave envelope gradient $\mathbf{F}_d' = \mathbf{F}_d = \mathbf{k}P_{rf}/\omega$. More generally, the eikonal limit of Eq. (7) yields the momentum moment of the wave-kinetic equation[8]

$$\frac{\partial}{\partial t}(\mathbf{k}N_k) + \nabla\cdot(\mathbf{v}_g\mathbf{k}N_k) + 2\gamma\mathbf{k}N_k + \nabla\omega\, N_k = \mathbf{F}_{ext} \qquad (11)$$

where $N_k = W_k/\omega$ is the wave action (W_k is the wave energy density), $\mathbf{v}_g$ is the group velocity and γ is the damping rate of the wave. The last two terms on the lhs of Eq. (11) can be derived from $\mathbf{F}_d'$.

In the full wave form [Eqs. (7) – (10)], wave-plasma momentum exchange can occur even for a cold-fluid plasma. Consider for example, the cold-fluid mode conversion from the fast wave to the ion cyclotron wave.[5,9] In the cold-fluid limit the $\mathbf{F}_d'$ term reduces to $-(\nabla\overleftrightarrow{\varepsilon}):\mathbf{EE}/8\pi$. We can interpret this force as compensating for mode-conversion or reflection-induced changes in the wave momentum flux term $\overleftrightarrow{T}_w$. Mode-conversion or reflection alone (without absorption) cannot drive flux-surface-averaged flows. However, this does not rule out radial forces on the plasma which would need to be balanced by corresponding $\mathbf{J}\times\mathbf{B}$ forces on the coils. In a reflection scenario, the wave momentum normal to the reflection surface is not conserved. Also, waves can be guided through a turn by a curved (e.g. poloidal) magnetic field. Again, the $\mathbf{F}_d'$ term compensates for the change in the wave momentum flux and ultimately represents a force that is transmitted to the supporting (coil) structures, analogous to the case of light being guided along a curved path by a fiber optic cable.

Finally, the integrated form of Eq. (5) may be useful for testing momentum conservation of the field solutions from full-wave rf codes,

$$\int d^2\mathbf{x} \cdot \left\langle \overleftrightarrow{T}_{em} \right\rangle + \int d^3x\, \mathbf{F}'_{pw} = \int d^3x\, \mathbf{F}_{ext} \qquad (12)$$

If the integration volume is bounded by a vacuum region in front of the antenna and walls, then $\mathbf{F}_{ext}$ = 0 and the $\overleftrightarrow{T}_{em}$ term describes the input of momentum. If the bounding surface is inside the antenna and walls, then $\overleftrightarrow{T}_{em}$ = 0, and the antenna-wall forces arise from the surface currents and charges by Eq. (10). If the bounding surface is in the vacuum or the walls, then the nonlinear stress term vanishes there (i.e. $\mathbf{F}_{pw}' = \mathbf{F}_{pw}$), and the net force exerted by the antenna and walls equals the net force on the plasma from direct wave momentum absorption.

CONCLUSIONS

We have shown that, as far as flux-surface-averaged toroidal and poloidal plasma flows are concerned, momentum is conserved by a system that consists of the plasma, the wave fields and the antenna-wall boundary conditions on the rf fields. In general, (i.e. not flux-surface averaged, or when considering radial momentum) the system must include the equilibrium magnetic field coils, because of induced $\mathbf{J} \times \mathbf{B}$ forces. These forces can be seen to arise in situations where waves or particles are guided by curved magnetic fields, or where waves undergo reflection or mode conversion. For flux-surface-averaged flows, the plasma and waves exchange momentum by direct absorption of wave momentum and by dissipative stress terms. An integrated form of the field momentum equation including rf forces on the antenna and walls should be useful in testing momentum conservation in full-wave rf codes, and relating it to the net rf force on the plasma that drives flux-surface-averaged flows.

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