Nonlinear fluxes and forces from radio-frequency waves with application to driven flows in tokamaks

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Abstract

Nonlinear, rf-driven sheared flows are of interest for turbulence control and basic physics experiments. Short wavelength slow modes are required for efficient coupling of wave momentum to the plasma, requiring a kinetic hot-plasma theory. Here, a guiding-center formulation is developed which calculates the nonlinear particle and energy fluxes, energy absorption and nonlinear forces on the plasma using a kinetic moment approach that is valid to first order in the ratio of the gyroradius compared to the wave envelope scale length and the plasma equilibrium scale length. Both the stress tensor and Lorentz force contribute to the net force on a fluid element. The forces driving flux-surface-averaged flows in a tokamak are extracted from the parallel and toroidal components. It is shown that flux-surface-averaged flows are driven by two classes of terms: direct absorption of wave momentum and dissipative stresses. Furthermore, the general kinetic expression for the force is shown to reduce to the standard cold-fluid ponderomotive force in an appropriate limit, but in this limit no flows are driven.

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I. Introduction

The importance of sheared plasma flows on plasma instabilities and the resulting turbulent transport is well established.[1] Spontaneously-occurring sheared flow layers are believed to be important in establishing the high confinement mode (H-mode) edge transport barrier and may play a role in some tokamak internal transport barriers. The prospect of externally controlling transport barrier formation has also been explored both experimentally and theoretically. Applied radio-frequency (rf) waves, especially in the ion cyclotron range of frequencies (ICRF) could, in principle, provide a flexible and practical means of external transport barrier control because of the ease with which rf power can be controlled and deposited at desired locations. This fact stimulated pioneering theoretical work in the area[2,3] followed by more recent calculations.[4-8] There are also a number of experimental results that seem to suggest ICRF or ion Bernstein wave (IBW) induced transport modifications and/or sheared plasma flows.[9-17] Theoretical and computational models for a conceptual understanding and quantitative analysis of these experiments, especially in two-dimensions (2D), have not been available.

Progress in numerical modeling of full wave solutions for ICRF waves is changing this situation. Two-dimensional full-wave solutions of mode conversion scenarios are becoming available.[18-20] In these mode conversion scenarios, short wavelength IBW and/or ion cyclotron waves (ICW) are produced,[19-22] and can be utilized for the non-linear generation of sheared flows. Short wavelength modes are required for efficient coupling of wave momentum to the plasma because the deposited momentum scales like $\mathbf{k}/\omega$ times the deposited wave energy. Short scale lengths are similarly favored for the bipolar sheared flow layers that are created by non-linear momentum redistribution, as has been investigated in References 4 – 7. While direct-launch of high-power IBWs has been achieved on some tokamaks,[12-16] it is of great interest to investigate the technologically easier possibility of using mode conversion to generate the short wavelength modes from fast waves. In addition to tools for performing the wave-field computations for these scenarios, a framework for modeling the resulting nonlinear rf forces is needed. This latter topic is the subject of the present paper where we develop a model that describes the (second order in wave amplitude) nonlinear response of a hot, weakly non-uniform plasma. In the process of obtaining the nonlinear momentum flux (require for the net force) we will also obtain expressions for the nonlinear particle and energy fluxes and the local energy absorption. Our expression for the local energy absorption will be seen to recover results obtained previously[6,23,24] in appropriate limits.

The present paper extends our recent work in this area by obtaining a compact form for the nonlinear rf-induced force on the plasma for tokamak geometry. The calculations extend the one-dimensional results of Refs. 5 and 6 to general geometry (with some assumptions) while generalizing the calculations of Ref. 7 from the eikonal limit. Motivated by the possibility of modeling flow drive experiments, we seek a result which expresses the desired nonlinear forces in a form suitable for implementation in spectral models such as the AORSA code[18] where the field quantities **E** are given in terms of a global Fourier expansion and the plasma dielectric properties are described in terms of the W matrix [see Appendix A]. Thus the present paper follows the formulation discussed in Ref. 6 while employing some of the calculational methods and compact dyad notation of Ref. 7. The form that we obtain for the nonlinear force shows a separation of reactive and dissipative terms (related to the anti-Hermitian and Hermitian parts of W, respectively). The former reduce to the well known cold fluid ponderomotive force in an appropriate limit, while the latter are shown to be capable of driving flux-surface-averaged flows. The separation of reactive and dissipative components of the force is a significant technical advance of the present paper.

The framework of our calculation is that of gyrokinetic theory ($k_\perp \rho_i \sim 1$ where ρ is the gyroradius) and ion-cyclotron frequency waves ($\omega \sim \Omega_i$) with resonant wave-particle interactions ($\omega - n\Omega_i \sim k_{||}v$) and electromagnetic plasma response. From our kinetic model, we calculate force densities on a fluid element that are suitable for use in transport models that describe the macroscopic evolution of a tokamak plasma. The force calculations are carried out to first order in ρ_i/L where L is an equilibrium or wave-envelope scale length.

The plan of our paper is as follows: In Sect. II we develop the moment equations for particle, energy and momentum conservation and delineate the quantities that are required from kinetic theory. It is shown that the nonlinear quantities separate into secular and non-secular terms that arise, respectively, from the gyrophase independent and gyrophase dependent parts of the distribution function. The secular terms are shown to be derivable from conventional (gyro-averaged) quasilinear theory, while the non-secular terms are the subject of the remainder of this paper. The kinetic guiding center formalism, a general treatment of non-secular moments, and the kinetic results required here are given in Sect III. In Section IV we collect together the results from the moment equations and kinetic calculations to obtain final expressions for the nonlinear forces and fluxes. In particular, the momentum moment is treated, including an examination of the Lorentz force terms and the nonlinear stress tensor. In Sect. V, we discuss the relationship of the present results to the conventional cold-fluid ponderomotive force. In Sect. VI, the forces which can drive flux-surface-averaged flows and radial electric fields are extracted from the general results of Sect. IV. This is followed by a discussion and summary of our conclusions in Sect. VII. The conceptual development of these topics is presented in the main text, but some of the calculations required to obtain explicit forms for the forces and fluxes in the Fourier basis are lengthy. These and various other technical details are relegated to appendices.

II. Moment equations

In lab coordinates ($\mathbf{r}$, $\mathbf{v}$), the Vlasov equation in conservative form is

$$\frac{\partial f}{\partial t} + \nabla \cdot (\mathbf{v} f) + \nabla_v \cdot (\Omega \mathbf{v} \times \mathbf{b} f) = -\nabla_v \cdot (\mathbf{a} f) \tag{1}$$

where $\mathbf{a}$ is the acceleration due to the rf fields, $f = f(\mathbf{r}, \mathbf{v})$ and ∇_v is at constant $\mathbf{r}$. Here it is useful to separate the rf forces contained in $\mathbf{a} = (Ze/m)(\mathbf{E}^{(1)} + \mathbf{v} \times \mathbf{B}^{(1)}/c)$ from the equilibrium Lorentz force on the lhs. Standard notations are employed for the charge Ze, mass m, wave frequency ω and gyro-frequency $\Omega = ZeB/mc$. We will occasionally use the notation $f^{(i,j)}$ where i refers to the order in rf electric field, and j refers to the order in ρ/L. When only one superscript appears, it refers to i, the electric field order. Using Maxwell's equations [and correcting a non-propagating sign error in Eq. (20) of Ref. 7] the rf electromagnetic acceleration can be rewritten as

$$\mathbf{a} = \frac{Ze}{m}\left[I\left(1 - \frac{\mathbf{k} \cdot \mathbf{v}}{\omega}\right) + \frac{\mathbf{k}\mathbf{v}}{\omega} \right] \cdot \mathbf{E}^{(1)} \ . \tag{2}$$

Taking moments in the lab frame, we have

$$\frac{\partial n}{\partial t} + \nabla \cdot \boldsymbol{\Gamma} = 0 \,, \tag{3}$$

$$\frac{\partial w}{\partial t} + \nabla \cdot \mathbf{Q}_m = \mathbf{J}^{(1)} \cdot \mathbf{E}^{(1)} \,, \tag{4}$$

$$m \frac{\partial \boldsymbol{\Gamma}}{\partial t} + \nabla \cdot \Pi = \frac{1}{c} \mathbf{J} \times \mathbf{B} + \mathbf{F}_L \,, \tag{5}$$

where

$$n = \int d^3 v \ f \ , \tag{6}$$

$$\mathbf{\Gamma} \equiv n\mathbf{u} = \int d^3v\ \mathbf{v}f\ , \tag{7}$$

$$w = \int d^3v\ \frac{1}{2}mv^2 f\ , \tag{8}$$

$$\mathbf{Q}_m = \int d^3v\ \frac{1}{2}mv^2 \mathbf{v}f\ , \tag{9}$$

$$\mathbf{J} \equiv Zen\mathbf{u}\ , \tag{10}$$

$$\Pi = m\int d^3v\ \mathbf{vv}f\ , \tag{11}$$

$$\mathbf{F}_L = Zen^{(1)}\mathbf{E}^{(1)} + \frac{1}{c}\mathbf{J}^{(1)} \times \mathbf{B}^{(1)}\ . \tag{12}$$

Note that the energy flux in Eqs. (4) and (9) is denoted $\mathbf{Q}_m$ (we will reserve the symbol $\mathbf{Q}$ for later use) and $\mathbf{B}$ in Eq. (5) is the equilibrium magnetic field.

We are interested in the time-averaged, nonlinear, order $|E|^2$ fluxes $<\mathbf{\Gamma}>_t$, $<\mathbf{Q}_m>_t$, and $<\Pi>_t$ driven by the rf waves, so henceforth, Eqs. (3) – (12) are understood to be time-averaged over an rf wave period, and where no confusion arises, the $<>_t$ notation for the time average will be suppressed. Thus for any nonlinear product AB, expressing A and B in terms of their Fourier representations, we will frequently abbreviate

$$AB =< AB >_t = \sum_{k,k'} \frac{1}{4} e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}} A_k^* B_{k'} + cc\,. \tag{13}$$

The Π and $\mathbf{F}_L$ terms in Eq. (5) will combine to give the total nonlinear force $\mathbf{F}$. If we assume ρ/L is a small parameter, where $\nabla \sim L^{-1}$ and L is the scale length of the equilibrium or the *envelope* of the rf waves, we require Γ, $\mathbf{Q}_m$, and Π to zero order in ρ/L to obtain $\mathbf{F}$ through first order. Note that while our calculation treats $\rho/L << 1$, it still permits $k\rho \sim 1$ where k is the rf wavenumber.

We observe that the gyroaveraged part of the distribution function is secular [since the rf can heat particles and there are no energy sinks in Eq. (1)] so it is useful to distinguish between the contributions to the moments that are driven by the secular (gyrophase averaged) and non-secular (gyrophase dependent) parts of f. In particular $\Gamma_{||}$, $Q_{||}$ and a piece of Π

$$\Pi_{cgl} = m\int d^3v\ \langle \mathbf{vv} \rangle_\phi\ f\ , \tag{14}$$

where $<>_\phi$ is a gyroaverage, are secular. The secular piece of the nonlinear stress tensor will be recognized as the Chew-Goldberger-Low (CGL) term $\Pi_{cgl} = (I - \mathbf{bb})\, p_\perp + \mathbf{bb}\, p_{||}$ where $\mathbf{b}$ is the unit vector along the equilibrium magnetic field. In the present work, we are interested in calculating the non-secular fluxes and forces, (the secular portions are obtained from conventional quasilinear evolution of $\langle f \rangle_\phi$) thus $\Gamma_{||}$, $Q_{||}$ and Π_{cgl} will be subtracted from the moments we calculate. To summarize, henceforth we concern ourselves with the calculation of

$$\mathbf{\Gamma}_\phi \equiv n\mathbf{u}_\perp = \int d^3v \ \mathbf{v}_\perp f^{(2,0)}, \tag{15}$$

$$\mathbf{Q}_\phi = \int d^3v \ \frac{1}{2} m v^2 \mathbf{v}_\perp f^{(2,0)}, \tag{16}$$

$$\Pi_\phi = m \int d^3v \ \left(\mathbf{vv} - \langle \mathbf{vv} \rangle_\phi\right) f^{(2,0)}, \tag{17}$$

to lowest order in ρ/L and the force on the plasma defined by

$$\mathbf{F} = -\nabla \cdot \Pi_\phi + \mathbf{F}_L, \tag{18}$$

and $\langle \mathbf{J} \cdot \mathbf{E} \rangle_t$ through first order in ρ/L.

The Lorenz force term $\mathbf{F}_L$ may be manipulated using Maxwell' s equations to give[7]

$$\mathbf{F}_L = \frac{1}{16\pi}\left[(\nabla \mathbf{E}^*) \cdot \mathbf{D} - \nabla \cdot (\mathbf{D}\mathbf{E}^*)\right] + cc. \tag{19}$$

Noting that in the Fourier representation $\nabla \mathbf{E}^* \rightarrow -i\mathbf{k}\mathbf{E}^*$ and $\mathbf{D} = 4\pi i \mathbf{J}/\omega$, both the momentum and energy equations have similar drive terms proportional to $\mathbf{J} \cdot \mathbf{E}$.

Consider the energy equation first. Equation (4) provides an expression for the heating rate and energy flux in terms of the distribution function. However, as has been noted before[23] the split-up between what is regarded as the heating rate and what contributes to the energy flux $\mathbf{Q}$ is arbitrary up to the divergence of a vector. Here, we make the split-up on the basis of the construction of the heating rate $\dot{w}$ as a symmetric (positive definite) bilinear form on the rf electric fields[23,24,25] [see Eq. (A22)] and absorb the remainder into the energy flux $\mathbf{Q}_w$. Namely,

$$\langle \mathbf{J} \cdot \mathbf{E} \rangle_t = \dot{w} - \nabla \cdot \mathbf{Q}_w. \tag{20}$$

The explicit expressions for $\mathbf{Q}_w$ and $\dot{w}$ will require a kinetic calculation. In Appendix A, $\dot{w}$ is defined by Eq. (A10) and $\mathbf{Q}_w$ by Eq. (A20). Combining with Eqs. (4) and (16) we have the energy conservation equation

$$\frac{\partial w}{\partial t} + \nabla \cdot \mathbf{Q} = \dot{w} \tag{21}$$

where

$$\mathbf{Q} = \mathbf{b} Q_{||} + \mathbf{Q}_\phi + \mathbf{Q}_w. \tag{22}$$

Parallel to the derivation of Eq. (20) from kinetic theory it will be shown in Sec III C and Appendix A that Eq. (19) can be recast using

$$\frac{1}{4\pi}\left\langle \nabla \mathbf{E}\cdot\mathbf{D}\right\rangle_t = \mathbf{F}_0 - \nabla\cdot\Pi_w \tag{23}$$

where for a wave spectrum with a single $\mathbf{k}$ component, $\mathbf{F}_0 = \mathbf{k}\dot{w}/\omega$ and $\Pi_w = \mathbf{k}\mathbf{Q}_w/\omega$. The general forms will be given subsequently. Thus the nonlinear force takes the form

$$\mathbf{F} = \mathbf{F}_0 - \nabla\cdot(\Pi_\phi + \Pi_{DE} + \Pi_w) \tag{24}$$

where from Eq. (19)

$$\Pi_{DE} = \frac{1}{4\pi}\left\langle \mathbf{D}\mathbf{E}^*\right\rangle_t . \tag{25}$$

In summary, the gyrophase-averaged distribution function $<f>_\phi$ obtained from conventional quasilinear calculations yields the quantities n, w, $\Gamma_{||}$, $Q_{||}$, Π_{cgl} while the gyrophase-dependent distribution function $f - <f>_\phi$ can be used to obtain non-secular results for $\mathbf{\Gamma}_\perp = \mathbf{\Gamma}_\phi$, $\mathbf{Q}_\perp = \mathbf{Q}_\phi + \mathbf{Q}_w$, $\dot{w}$ and $\mathbf{F}$. Thus we require kinetic calculations for the moments $\mathbf{\Gamma}_\phi$, $\mathbf{Q}_\phi$ and Π_ϕ [Eqs. (15) – (17)] and for the explicit forms of Eqs. (20) and (23). It is important to emphasize that the moments $\mathbf{\Gamma}_\phi$, $\mathbf{Q}_\phi$ and Π_ϕ are only required to lowest order in ρ/L while for Eqs. (20) and (23) the linearized current $\mathbf{J}$ (or more specifically $\mathbf{J}\cdot\mathbf{E}$) is required through first order in ρ/L.

III. Kinetic calculations

A. Guiding center formalism

The transformation from lab coordinates ($\mathbf{r}$, $\mathbf{v}$) to guiding center coordinates ($\mathbf{R}$, $v_\perp$, $v_{||}$, ϕ) is given by

$$\mathbf{R} = \mathbf{r} + \frac{1}{\Omega}\mathbf{v}\times\mathbf{b} , \tag{26}$$

$$\mathbf{v} = v_\perp(\mathbf{e}_x \cos\phi + \mathbf{e}_y \sin\phi) + v_{||}\mathbf{b} . \tag{27}$$

Here $\mathbf{e}_x$ and $\mathbf{e}_y$ are arbitrary orthogonal coordinates normal to the local magnetic field $\mathbf{b}$ and form a local Stix coordinate system. Noting that

$$\frac{\partial\mathbf{R}}{\partial\mathbf{v}} = \frac{1}{\Omega}\mathrm{I}\times\mathbf{b} , \tag{28}$$

and that $\mathrm{I}\times\mathbf{b}\cdot\mathbf{a} = \mathbf{b}\times\mathbf{a}$ the Vlasov equation for f($\mathbf{R}$, $\mathbf{v}$) becomes

$$\frac{\partial f}{\partial t} + v_{\parallel}\nabla_{\parallel} f - \Omega \frac{\partial f}{\partial \phi} = -\mathbf{a}\cdot\nabla_v f - \frac{1}{\Omega}\mathbf{a}\cdot\mathbf{b}\times\nabla f\ , \tag{29}$$

or in conservative form

$$\frac{\partial f}{\partial t} + \nabla_{\parallel}(v_{\parallel} f) - \Omega \frac{\partial f}{\partial \phi} = -\nabla_v \cdot (\mathbf{a} f) - \nabla\cdot\left(\frac{1}{\Omega}\mathbf{b}\times\mathbf{a}\ f\right), \tag{30}$$

where ∇_v is at constant **R**. Equation (29) is useful for calculating the linearized current. For the nonlinear time-averaged calculations of $\mathbf{\Gamma}_{\perp}$, $\mathbf{Q}_{\perp}$ and Π Eq. (30) is more convenient, and for these lowest order calculations the last term on the rhs may be omitted resulting in

$$-\Omega \frac{\partial \tilde{f}^{(2)}}{\partial \phi} = -\nabla_v \cdot \left\langle \mathbf{a}^{(1)} f^{(1)} \right\rangle_t \tag{31}$$

for the gyrophase-dependent part of f given by

$$f = \langle f \rangle_{\phi} + \tilde{f}\ . \tag{32}$$

Here we have also used the fact that to lowest order in ρ/L the time-averaged nonlinear distribution function has $\partial/\partial t << \Omega$ and $v_{\parallel}\nabla_{\parallel} << \Omega$. Where no confusion can arise, we will frequency omit the super tilde ~ on f in the following.

Regarding the $v_{\parallel}\nabla_{\parallel} << \Omega$ assumption, the parallel flux is taken to be sufficiently large that it will rapidly smooth any spatial dependence in the parallel direction for the second order distribution function. Somewhat equivalently, we will only want flux surface averages. Thus we can drop the convective term on the lhs and assume that we have a Kronecker-delta in the parallel wave vector that results in only a single sum over parallel wave vectors.

Finally, we digress momentarily to consider the quasilinear time and gyro-averaged kinetic equation

$$\left(\frac{\partial}{\partial t} + v_{\parallel}\nabla_{\parallel}\right)\left\langle f^{(2)} \right\rangle_{t,\phi} = -\left\langle \nabla_v \cdot (\mathbf{a}^{(1)} f^{(1)}) \right\rangle_{t,\phi} - \nabla\cdot\left\langle \frac{1}{\Omega}\mathbf{b}\times\mathbf{a}^{(1)}\ f^{(1)} \right\rangle_{t,\phi} . \tag{33}$$

This is the conventional quasilinear equation from which the secular portions of f are obtained. For present purposes we only wish to note that there is a flux of guiding centers given by the last term, which will be useful in interpreting some later results.

B. Evaluation of nonlinear moments

Consider the X moment of the nonlinear time-average gyrophase-dependent function $<\tilde{f}^{(2)}>_t$ where $X = \mathbf{v}_{\perp}$, $½\, mv^2\, \mathbf{v}_{\perp}$, $\mathbf{vv} - <\mathbf{vv}>_{\phi}$,

$$L(X) = \int d^3v \left(X - \langle X \rangle_\phi \right) f^{(2)} , \tag{34}$$

and where $f^{(2)}$ obeys Eq. (31). Note that any gyrophase independent terms in $f^{(2)}$ will not contribute to L. Let

$$M = \int^{\phi} d\phi \left(X - \langle X \rangle_\phi \right) \tag{35}$$

where the constant of integration is chosen so that $<M>_\phi = 0$. It follows that

$$L = \int d^3v \frac{\partial M}{\partial \phi} f^{(2)} = -\int d^3v M \frac{\partial f^{(2)}}{\partial \phi} . \tag{36}$$

Using Eq. (31) and integrating by parts in **v** we obtain

$$L = -\frac{1}{\Omega} \int d^3v M \nabla_v \cdot (\mathbf{a} f^{(1)}) = \frac{1}{\Omega} \int d^3v f^{(1)} \mathbf{a} \cdot \nabla_v M . \tag{37}$$

Thus, we have shown that

$$\int d^3v \left(X - \langle X \rangle_\phi \right) f^{(2)} = \frac{1}{\Omega} \int d^3v < f^{(1)} \mathbf{a} >_t \cdot \nabla_v M \tag{38}$$

for any X where M is given by Eq. (35). Since **a** is first order in the rf field, Eq. (38) permits the calculation of moments of the *nonlinear* distribution function $\tilde{f}^{(2)}$ in terms of moments of the *linear* distribution function $f^{(1)}$. An explicit calculation of the nonlinear $\tilde{f}^{(2)}$ is therefore not required. This greatly simplifies the algebra over previous work,[5] but in the same limits we will find equivalent results. Furthermore, it will turn out that the required linear moments can all be expressed in terms of the linearized current **J**, or equivalently in terms of a quantity W that is closely related to the usual linear conductivity tensor.

For the nonlinear particle flux we have $X = \mathbf{v}_\perp$, $M = \mathbf{v} \times \mathbf{b}$, $\nabla_v M = I \times \mathbf{b}$ and therefore

$$\mathbf{\Gamma}_\phi = \left\langle \frac{1}{\Omega} \int d^3v f^{(1)} \mathbf{a} \times \mathbf{b} \right\rangle_t . \tag{39}$$

For the nonlinear energy flux we have $\mathbf{X} = ½\, mv^2\, \mathbf{v}_\perp$, $\mathbf{M} = ½\, mv^2\, \mathbf{v} \times \mathbf{b}$, $\nabla_v \mathbf{M} = ½\, mv^2\, I \times \mathbf{b} + m\mathbf{v}\mathbf{v} \times \mathbf{b}$ so that

$$\mathbf{Q}_\phi = \left\langle \frac{m}{\Omega} \int d^3v f^{(1)} \left(\frac{v^2}{2} \mathbf{a} \times \mathbf{b} + \mathbf{a} \cdot \mathbf{v}\, \mathbf{v} \times \mathbf{b} \right) \right\rangle_t . \tag{40}$$

Finally, for the nonlinear stress tensor we have $\mathbf{X} = \mathbf{v}\mathbf{v} - <\mathbf{v}\mathbf{v}>_\phi$,

$$M = \frac{1}{4} (\mathbf{v}_\perp\, \mathbf{v} \times \mathbf{b} + \mathbf{v} \times \mathbf{b}\ \mathbf{v}_\perp) + (\mathbf{v}_\parallel\, \mathbf{v} \times \mathbf{b} + \mathbf{v} \times \mathbf{b}\ \mathbf{v}_\parallel) , \tag{41}$$

yielding

$$\Pi_\phi = \left\langle \frac{m\mu}{\Omega} \int d^3v\, f^{(1)} \, (\mathbf{a}\,\mathbf{v}\times\mathbf{b} + \mathbf{v}\,\mathbf{a}\times\mathbf{b}) \right\rangle_t + \mathrm{tr}\,, \qquad (42)$$

where tr indicates the transpose of the immediately preceding term and $\mu = 1/4$ for the perpendicular components, Π_{ij} (i, j = 1, 2) and $\mu = 1$ for the perpendicular-parallel components (i = 3, j = 1, 2 or j = 3, i = 1, 2).

C. Results for $<J\cdot E>_t$ and $<\nabla E\cdot D>_t$

To calculate the rhs of the energy and momentum equations we require kinetic evaluations of the terms $<\mathbf{J}\cdot\mathbf{E}>_t$ and $<\nabla\mathbf{E}\cdot\mathbf{D}>_t$ through first order in ρ/L. We begin with the linearized Vlasov equation in guiding center variables

$$\frac{\partial f^{(1)}}{\partial t} + v_\parallel \nabla_\parallel f^{(1)} - \Omega \frac{\partial f^{(1)}}{\partial \phi} = -\mathbf{a}\cdot\nabla_v f_M - \frac{1}{\Omega}\mathbf{a}\cdot\mathbf{b}\times\nabla f_M \equiv S(\mathbf{R}, v_\perp v_\parallel, \phi) \qquad (43)$$

where the lowest order distribution function has been taken to be a local Maxwellian. The source term on the rhs is first expressed as a function of the guiding center variables. Solving Eq. (43) by standard techniques one obtains $f^{(1)}$ ($\mathbf{R}$, $v_\perp$,$v_\parallel$, ϕ). From this one can evaluate the current

$$\mathbf{J} = Ze\int d^3v \;\; \mathbf{v} \;\; f^{(1)} \qquad (44)$$

and the time-averaged heating source

$$\langle \mathbf{J}\cdot\mathbf{E}\rangle_t = \left\langle m\int d^3v \;\; \mathbf{v}\cdot\mathbf{a} \;\; f^{(1)} \right\rangle_t , \qquad (45)$$

where in evaluating the velocity integral, **r** not **R** must be held fixed.

Working through first order in ρ/L we can write

$$\langle \mathbf{J}\cdot\mathbf{E}\rangle = \dot{w} - \nabla\cdot\mathbf{Q}_w \qquad (46)$$

where the leading order term is

$$\dot{w} = \left\langle m\int d^3v \left[\mathbf{v}\cdot\mathbf{a}\; f^{(1)}\right]_{\mathbf{R}=\mathbf{r}} \right\rangle_t \equiv \langle \mathbf{E}\cdot W\cdot\mathbf{E}\rangle_t \qquad (47)$$

and the first order correction in the inverse guiding center transformation ~ $\Omega^{-1}\mathbf{b}\times\mathbf{v}\cdot\nabla$ is

$$\mathbf{Q}_w = \left\langle \frac{m}{\Omega}\int d^3v\, \mathbf{b}\times\mathbf{v}\;\; \mathbf{v}\cdot\mathbf{a}\;\; f^{(1)} \right\rangle_t . \qquad (48)$$

This result is the motivation for the particular separation of $\langle \mathbf{J}\cdot\mathbf{E}\rangle$ into energy and energy flux. It is shown in Appendix A that W is the symmetric, positive definite bilinear local energy absorption operator defined in Ref. 6 and discussed in Refs. 23 and 24. As noted by Smithe,[23] W generalizes the conductivity tensor: in the Fourier representation $W(\mathbf{k}, \mathbf{k}')$ reduces to $\sigma(\mathbf{k}) = W(\mathbf{k}, \mathbf{k})$ in the limit of equal arguments.

The calculation of $\langle \nabla\mathbf{E}\cdot\mathbf{D}\rangle_t$ is similar to that of $\langle \mathbf{J}\cdot\mathbf{E}\rangle_t$, the only difference being that there is an extra ∇ acting on the $\mathbf{E}$ in Eq. (45). The result is an immediate generalization of Eqs. (46) - (48). Thus we have

$$\left\langle \frac{1}{4\pi}(\nabla\mathbf{E})\cdot\mathbf{D}\right\rangle_t = \mathbf{F}_0 - \nabla\cdot\Pi_w , \tag{49}$$

where the lowest order force in the guiding center expansion is

$$\mathbf{F}_0 = \left\langle \left(\frac{\mathbf{k}}{\omega}\mathbf{E}\right)\cdot W\cdot\mathbf{E}\right\rangle_t , \tag{50}$$

and the first order correction is

$$\Pi_w = \left\langle \frac{m}{\Omega}\int d^3v\, f^{(1)}\mathbf{b}\times\mathbf{v}\frac{\mathbf{k}}{\omega}\mathbf{a}\cdot\mathbf{v}\right\rangle_t . \tag{51}$$

Here we define the operator

$$\frac{\mathbf{k}}{\omega} = \frac{1}{\omega^2}\frac{\partial}{\partial t}\nabla . \tag{52}$$

More details of the calculation of $\mathbf{Q}_w$ and Π_w are in Appendix A where the results are given explicitly in the Fourier representation for a hot plasma. By manipulating Eq. (50) into a form where ∇ operates symmetrically with respect to the $\mathbf{E}$ on the left of W and the one on the right, the dissipative and reactive parts, respectively $\mathbf{F}_{0d}$ and $\mathbf{F}_r$ emerge:

$$\mathbf{F}_0 = \mathbf{F}_{0d} + \mathbf{F}_r . \tag{53}$$

It is shown in Appendix B that in the Fourier representation

$$\mathbf{F}_{0d} = \frac{1}{4\omega}\sum_{kk'} e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}}(\mathbf{k}+\mathbf{k}')\mathbf{E}_k^*\cdot H(\mathbf{k},\mathbf{k}')\cdot\mathbf{E}_{k'} , \tag{54}$$

$$\mathbf{F}_r = \frac{1}{4\omega}\sum_{kk'} e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}}(\mathbf{k}-\mathbf{k}')\mathbf{E}_k^*\cdot A(\mathbf{k},\mathbf{k}')\cdot\mathbf{E}_{k'} , \tag{55}$$

where H and A are the Hermitian and anti-Hermitian parts of W. The combination $\mathbf{k}-\mathbf{k}'$ appearing in the definition of $\mathbf{F}_r$ implies a gradient on the envelope scale length, thus making $\mathbf{F}_{0d}$ formally zero-order in the ρ/L expansion while $\mathbf{F}_r$ is first order.

IV. Nonlinear forces and fluxes

A. Particle flux and ambipolarity

Collecting terms together, we can now examine the moment equations in more detail. Particle conservation is given by Eqs. (3) and (39). We can write the perpendicular particle flux as

$$\mathbf{\Gamma}_\perp = -\frac{1}{\Omega}\mathbf{b}\times\left\langle \int d^3v\, f^{(1)}\,\mathbf{a}\right\rangle_t = -\frac{1}{m\Omega}\mathbf{b}\times\mathbf{F}_L\,, \tag{56}$$

where only the lowest order part of $\mathbf{F}_L$ is required here, and is given by $\mathbf{F}_{0d}$. Thus the particle flux is just the guiding center drift due to the lowest order force that arises from direct wave momentum absorption. In general, when summed over species, the electron and ion particle currents will not cancel, and result in a charge source proportional to

$$\nabla\cdot\mathbf{J} = \sum_{i,e}\frac{c}{B}\mathbf{b}\cdot\nabla\times\mathbf{F}_{0d}\,. \tag{57}$$

This net force on the plasma and its resulting charge imbalance acts as a source term in the vorticity equation and generates an ambipolar potential Φ.

B. Energy flux and heating rate

The energy equation is from Eq. (21) and following (repeated here for convenience)

$$\frac{\partial w}{\partial t} + \nabla\cdot\mathbf{Q} = \dot{w} \tag{58}$$

where $\mathbf{Q} = \mathbf{Q}_{||} + \mathbf{Q}_\perp$ and $\mathbf{Q}_\perp = \mathbf{Q}_\phi + \mathbf{Q}_w$. Combining Eqs. (40) and (48), there is a cancellation of terms resulting in

$$\mathbf{Q}_\perp = \left\langle \frac{m}{2\Omega}\int d^3v\, f^{(1)}\, v^2\mathbf{a}\times\mathbf{b}\right\rangle_t\,. \tag{59}$$

Thus the energy flux integral is analogous to the particle flux integral with an additional factor of $\frac{1}{2}mv^2$. Comparing the form of both fluxes with the last term in the gyro-averaged kinetic equation, Eq. (33), we see that these fluxes can be interpreted as guiding center drifts.

C. Total nonlinear force

The momentum equation takes the form

$$m \frac{\partial \mathbf{\Gamma}}{\partial t} + \nabla \cdot \Pi_{cgl} - \frac{1}{c} \mathbf{J} \times \mathbf{B} = \mathbf{F}, \tag{60}$$

where the nonlinear stress tensor and Lorentz force terms combine to yield the total nonlinear force given by Eq. (24), viz.

$$\mathbf{F} = \mathbf{F}_0 - \nabla \cdot (\Pi_\phi + \Pi_w + \Pi_{DE}), \tag{61}$$

where $\mathbf{F}_0$ is given by Eqs. (53) – (55), Π_ϕ by Eq. (42), Π_w by Eq. (51), and Π_{DE} is given by Eq. (25). We now seek a more compact form of the combined tensor terms under the divergence.

In Appendix B.2 it is shown that the *linearized* momentum equation can be used to express Π_{DE} in an alternative form. The result is

$$\Pi_{DE} = \left\langle \frac{Ze}{\Omega} \int d^3v\, f^{(1)} \mathbf{b} \times \mathbf{v} \left(1 - \frac{\mathbf{k} \cdot \mathbf{v}}{\omega} \right) \mathbf{E} + \frac{1}{4\pi} \mathbf{D}_{\|} \mathbf{E} \right\rangle_t , \tag{62}$$

where some terms that ultimately vanish under the assumption of species-summed quasineutrality have not been retained. Combining the preceding expression with Eq. (51) yields

$$\Pi_{DE} + \Pi_w = \left\langle \frac{m}{\Omega} \int d^3v\, f^{(1)}\, \mathbf{b} \times \mathbf{v}\, \mathbf{a} + \frac{1}{4\pi} \mathbf{D}_{\|} \mathbf{E} \right\rangle_t . \tag{63}$$

Restricting the derivation for the remainder of this section to the perpendicular plane (i.e. the force in the perpendicular direction due to perpendicular field gradients) we can drop the $D_{\|}E$ term and combine the preceding result with Π_ϕ to obtain

$$\Pi_\phi + \Pi_w + \Pi_{DE} = \left\langle \frac{m}{\Omega} \int d^3v\, f^{(1)} \left(\frac{1}{4} (\mathbf{a}_\perp \mathbf{v} + \mathbf{v}_\perp \mathbf{a}) \times \mathbf{b} + \mathrm{tr} + \mathbf{b} \times \mathbf{v}\, \mathbf{a}_\perp \right) \right\rangle_t . \tag{64}$$

The following identities are useful

$$\frac{1}{4} \left[(\mathbf{a}_\perp \mathbf{v} + \mathbf{v}_\perp \mathbf{a}) \times \mathbf{b} + \mathrm{tr} \right] - \mathbf{v} \times \mathbf{b}\, \mathbf{a}_\perp = \frac{1}{2} I_\perp \mathbf{b} \cdot \mathbf{v} \times \mathbf{a} + \frac{1}{2} I \times \mathbf{b}\, \mathbf{v}_\perp \cdot \mathbf{a}_\perp , \tag{65}$$

$$\nabla \cdot \left(I_\perp \mathbf{b} \cdot \mathbf{v} \times \mathbf{a} + I \times \mathbf{b}\, \mathbf{v}_\perp \cdot \mathbf{a}_\perp \right) = \nabla_\perp (\mathbf{b} \cdot \mathbf{v} \times \mathbf{a}) - \mathbf{b} \times \nabla (\mathbf{v}_\perp \cdot \mathbf{a}_\perp), \tag{66}$$

and result in

$$\mathbf{F} = \mathbf{F}_{0d} + \mathbf{F}_r - \nabla_\perp X_r + \mathbf{b} \times \nabla X_d , \tag{67}$$

where

$$X_r = \frac{m}{2\Omega} \left\langle \int d^3v\, \mathbf{b} \cdot \mathbf{v} \times \mathbf{a}\, f^{(1)} \right\rangle_t \tag{68}$$

and

$$X_d = \frac{m}{2\Omega}\left\langle \int d^3v\ \mathbf{v}_\perp \cdot \mathbf{a} f^{(1)} \right\rangle_t \equiv \frac{\dot{w}_\perp}{2\Omega}. \tag{69}$$

If the elementary single particle force $m\mathbf{a}_\perp$ is thought of as consisting of components perpendicular and parallel to $\mathbf{v}_\perp$, i.e. $\mathbf{a}_\perp = (\mathbf{a}_\perp \cdot \mathbf{v}_\perp + \mathbf{a}\cdot\mathbf{b}\times\mathbf{v})/v_\perp$ then it will be seen that X_d is a measure of the perpendicular wave-particle energy exchange to wave dissipation (weighted by the linear distribution function) and X_r is a measure of the reactive component of the wave-particle energy which will be seen to be associated with internal magnetization. It is shown in Appendix C that X_r and X_d have compact expressions in terms of W:

$$X_d = \frac{1}{2\omega}\left\langle \sum_n n\mathbf{E}\cdot W_n \cdot \mathbf{E} \right\rangle_t , \tag{70}$$

$$X_r = \frac{1}{2\omega^2}\left\langle \frac{\partial \mathbf{E}}{\partial t}\cdot W \cdot \mathbf{E} + k_\perp \frac{\partial}{\partial k_\perp}\frac{\partial \mathbf{E}}{\partial t}\cdot W \cdot \mathbf{E} \right\rangle_t , \tag{71}$$

where in X_d, W has been separated into its gyro-resonant ($\omega - k_\parallel v_\parallel = n\Omega$) components W_n and in X_r the k derivative is to be taken on the wavevector associated with the E on the left, i.e. on the first argument of $W(k, k')$ (see Appendix C).

Flux-surface-averaged flows will be seen to result from X_d but not X_r. In Sect. V and Appendix D the reactive terms X_r and $\mathbf{F}_r$ are shown to be closely related to the cold fluid ponderomotive force.[26]

The formalism allows computation of forces in both the perpendicular and parallel directions due to both perpendicular and parallel field gradients; however, the result is sufficiently complicated that it sheds little insight. For most situations of interest for flow drive in tokamaks, off-diagonal elements $\Pi_{\perp,\parallel}$ and $\Pi_{\parallel,\perp}$ do not make substantial contributions. It will be shown in Sec. VI that the term $\Pi_{\parallel,\parallel}$ as well as the diagonal elements of $\Pi_{\perp,\perp}$ (i.e. Π_{cgl}) do not provide torques that drive flux surface-averaged flows.

V. Relation to the cold fluid ponderomotive force

As a check on the present calculations, it is useful to show that the nonlinear force derived here reduces to the conventional fluid ponderomotive force in an appropriate limit. The details of this demonstration are lengthy and may be found in Appendix D. Here we wish to make the correspondence plausible, and discuss the implications for flow drive calculations.

In comparing the cold-fluid limit of the present calculation with the conventional ponderomotive force,[26] it is necessary to recall that in the derivation of the nonlinear stress tensor, we omitted terms of the CGL type which are secular in kinetic theory (i.e. they are secular if the equations allow dissipation of wave energy into plasma heating but do not include sink terms such as transport to remove the heat). In making the comparison with the conventional cold-fluid ponderomotive force, they must be retained.

A general form for the cold-fluid ponderomotive force is given by[26,27]

$$\mathbf{F} = -n\nabla\psi_p + \mathbf{B}\times\nabla\times\mathbf{M} \tag{72}$$

where the ponderomotive potential ψ and magnetization **M** (no relation to the scalar M temporarily introduced in Sec. III B) are given by

$$\psi_p = \frac{iZe}{8\omega}\mathbf{u}^*\cdot\mathbf{E} + cc = -\frac{1}{32\pi n}\mathbf{E}^*\cdot\mathbf{D} + cc\,, \tag{73}$$

$$\mathbf{M} = -\frac{iZen}{8\omega c}\mathbf{u}^*\times\mathbf{u} + cc\,, \tag{74}$$

and **u** is given as the solution of the cold fluid momentum equation

$$-i\omega\mathbf{u} - \Omega\mathbf{u}\times\mathbf{b} = \frac{Ze}{m}\mathbf{E} \equiv \mathbf{a}\,. \tag{75}$$

Note that in the fluid limit, the perturbed magnetic field need not be retained in the perturbed acceleration **a**.

First, considering the perpendicular force from perpendicular gradients in the limit where the equilibrium magnetic field is constant, we have

$$\mathbf{F} = -n\nabla\psi + \nabla_\perp(BM_\parallel)\,. \tag{76}$$

This can be compared to the cold-fluid limit of Eq. (67), viz.

$$\mathbf{F}' = \mathbf{F}_r - \nabla_\perp X_r\,, \tag{77}$$

where the prime denotes that we have yet to add the CGL terms. In Eq. (77) only reactive terms are retained and in the fluid limit we note that W is independent of **k** so that W·**E** → **J** = nm**u**.

In the cold-fluid limit, the total nonlinear stress tensor (including Π_{cgl}) is

$$\Pi = nm\mathbf{uu}\,. \tag{78}$$

Thus, adding the CGL term to Eq. (77) we obtain

$$\mathbf{F} = \mathbf{F}_r - \nabla_\perp\left(X_r + \frac{1}{2}nm\left\langle u_\perp^2\right\rangle_t\right), \tag{79}$$

$$\mathbf{F}_r = \frac{1}{16\pi}(\nabla\mathbf{E}^*)\cdot\mathbf{D} + \text{cc}\,, \tag{80}$$

$$X_r = \frac{nm}{8\Omega}\mathbf{b}\cdot\mathbf{u}\times\mathbf{a}^* + \text{cc}\,. \tag{81}$$

The equivalence of Eqs. (76) and (79) can be shown after some algebra. In this simple limit where the spatial variation of the dielectric susceptibility χ (where $\mathbf{D} = \chi\cdot\mathbf{E}$) is only through the density n, the term $\mathbf{F}_r$ maps directly to $-n\nabla\psi_p$ while the remaining term maps to $\nabla_\perp(BM_{||})$.

It is significant that Eq. (72) holds in general magnetic geometry (i.e. accounting for ∇ operating on $\mathbf{b}$). In this case, we can show the equivalence of

$$\mathbf{F} = \frac{1}{16\pi}\left[(\nabla\mathbf{E}^*)\cdot\mathbf{D} - \nabla\cdot(\mathbf{D}\mathbf{E}^* + 4\pi nm\mathbf{u}\mathbf{u}^*)\right] + \text{cc} \tag{82}$$

with Eq. (72). The $\nabla\mathbf{E}\cdot\mathbf{D}$ term differs from the $-n\nabla\psi_p$ term by $\nabla(\chi/n)$. Remarkably, these $\nabla(\chi/n)$ terms are exactly what is required to account for the operations of ∇ on $\mathbf{b}$ in transforming the divergence term in Eq. (82) into the magnetization form of Eq. (72).

The kinetic derivation presented in this paper has not retained $\nabla\mathbf{b}$ equilibrium terms. The significance of the form of Eq. (72) is that in the next section it will allow us to show that the reactive cold-fluid terms in the nonlinear force cannot generate any flux-surface-averaged flows, even retaining $\nabla\mathbf{b}$ effects.

VI. Flux-surface-averaged flows and the radial electric field

Nonlinear rf forces can drive flux-surface-averaged flows in a tokamak and modify the radial electric field. The following subsections consider these effects.

A. Flux-surface-averaged flows

For flux-surface-averaged flows in a tokamak, we are concerned with forces and flows *in* the flux surface, namely poloidal and toroidal. The two components of flow in the surface are best expressed in terms of projections onto the parallel $\mathbf{b}$ and toroidal $\mathbf{e}_\zeta$ directions.[28] The relevant neoclassical viscosity[29,30] (which damps poloidal flows) enters most naturally from the parallel momentum equation while the toroidal angular momentum equation is convenient in a torus because of axisymmetry. The necessary averages of the rf force for describing flux-surface-averaged flows are $\langle BF_{||}\rangle_\psi$ and $\langle RF_\zeta\rangle_\psi$ where $\langle\ldots\rangle_\psi$ is a flux-surface average[28] (see Appendix E).

We first show that the reactive cold-fluid ponderomotive force given by Eq. (72) cannot drive any flux-surface-averaged flows in a tokamak. The toroidal component is given by

$$\left\langle R\mathbf{e}_\zeta \cdot \mathbf{F} \right\rangle_\psi = -n\left\langle R\mathbf{e}_\zeta \cdot \nabla\psi_p \right\rangle_\psi + \left\langle R\mathbf{e}_\zeta \cdot \mathbf{B}\times\nabla\times\mathbf{M} \right\rangle_\psi \tag{83}$$

where we have used the fact that n = n(ψ) is a flux function. [The equilibrium magnetic flux function ψ should not be confused with the ponderomotive potential ψ_p in this discussion.] The first term in Eq. (83) vanishes immediately since $R\mathbf{e}_\zeta \cdot \nabla = \partial/\partial\zeta$ and the flux surface average introduces $\int d\zeta$. For the second term, we interchange dot and cross and employ the representation

$$\mathbf{B} = B_\zeta \mathbf{e}_\zeta + \nabla\zeta\times\nabla\psi \tag{84}$$

so that $R\mathbf{e}_\zeta \times \mathbf{B} = -\nabla\psi$ and Eq. (83) becomes

$$\begin{aligned} \left\langle R\mathbf{e}_\zeta \cdot \mathbf{F} \right\rangle_\psi &= -\left\langle \nabla\psi\cdot\nabla\times\mathbf{M} \right\rangle_\psi = \left\langle \nabla\cdot(\nabla\psi\times\mathbf{M}) \right\rangle_\psi \\ &= \frac{1}{\upsilon}\frac{\partial}{\partial\psi}\upsilon\left\langle \nabla\psi\cdot\nabla\psi\times\mathbf{M} \right\rangle_\psi = 0 \quad . \end{aligned} \tag{85}$$

Next we examine the averaged parallel force component

$$\left\langle \mathbf{B}\cdot\mathbf{F} \right\rangle_\psi = -n\left\langle \mathbf{B}\cdot\nabla\psi_p \right\rangle_\psi = 0 , \tag{86}$$

where we have again used n = n(ψ) and the identity that $\left\langle B\nabla_{\|} Y \right\rangle_\psi = 0$ for any scalar Y.

It is theoretically possible that in a full kinetic treatment of χ that retains $\nabla\mathbf{b}$ terms, some reactive FLR terms could, in the presence of curvilinear magnetic fields, produce a flow drive force. However, not only does this seem unlikely, (since all the other flow drive force terms are dissipative) but such terms would be numerically small, depending not on the gradient of $|E|^2$ or even on density and temperature gradients, but on the magnetic field gradient which is order 1/R. In any case, such terms are outside the scope of the present calculation.

Thus the flow drive forces result from the $\langle BF_{\|}\rangle_\psi$ and $\langle RF_\zeta\rangle_\psi$ averages of the dissipative parts of Eq. (67), namely

$$\mathbf{F}_{dis} = \mathbf{F}_{0d} + \mathbf{b}\times\nabla X_d . \tag{87}$$

Using the identities given in Appendix E it can be shown directly that the remaining terms make no contribution to the flux-surface averages when magnetic field gradient terms of order 1/R are neglected.

The first term, F_{0d}, reduces to $\dot{w}\,\mathbf{k}/\omega$ in the eikonal limit, and is easily recognized as the momentum **k** absorbed by the plasma per "photon" i.e. per unit of absorbed wave action, $\dot{w}/\omega$. The second term in Eq. (87) is the two-dimensional covariant form of the stress term treated in our earlier publications.[6,7] Thus flux-surface-averaged flows are driven by two classes of terms: direct absorption of wave momentum and dissipative stresses. The former can drive net flows, while the latter imply a redistribution of plasma momentum (without net momentum input) and drive bipolar sheared flows. We note that these two classes of rf flow-drive terms were first identified in Ref. 3, in a more idealized analysis that did not treat two-dimensional tokamak geometry or the full kinetic plasma response.

B. Radial electric field

The modifications to the radial electric field, are of particular interest for applications to turbulence suppression. The radial electric field may be obtained from the ion radial force balance equation,

$$G = -c\left(\frac{\partial\Phi}{\partial\psi} + \frac{1}{Zen}\frac{\partial p_i}{\partial\psi}\right) + \left\langle \frac{c}{Zen_i}\frac{F_{i\psi}}{RB_\theta}\right\rangle_\psi , \tag{88}$$

where $G(\psi)$, $\Phi(\psi)$, and $p_i(\psi)$ are flux functions to the order considered, and the flow representation,

$$\mathbf{u} = K(\psi)\mathbf{B} + G(\psi)R\mathbf{e}_\zeta \tag{89}$$

identifies G as the toroidal rotation. The rf waves can affect the radial electric field (and its shear) in three ways. The rf heating can modify the pressure profile p_i, the rf forces modify the plasma flow G¸ and the radial rf forces contribute directly to the electric field through the $F_{i\psi}$ term in Eq. (88).

VII. Discussion and Conclusions

In this paper we have developed a hot plasma, weakly nonlocal, electromagnetic theory of the nonlinear forces and fluxes induced by rf waves on a two-dimensional plasma. The particle and energy fluxes are expressible in terms of guiding center drifts caused by the nonlinear Lorentz force on the plasma, and the local energy absorption can be expressed in terms of an operator $W(\mathbf{k}, \mathbf{k}')$, familiar from previous calculations, which generalizes the conductivity tensor $\sigma(\mathbf{k})$. The wave-induced forces on the plasma have been obtained by starting from the nonlinear Lorentz force and nonlinear stress terms given by Eqs. (12) and (17). It was shown that a final expression for the total nonlinear force on the plasma could also be obtained in terms of W. The guiding-center formalism on which this analysis based allows calculations of all relevant quantities without the need to obtain the second-order, quasilinear distribution function and provides physical insights for the plasma response.

The present work extends previous calculations to two-dimensional geometry using a spectral representation that is appropriate for use in conjunction with full wave codes that employ Fourier basis functions. Finite Larmor radius expansions (although with substantial algebra) can be derived from the Fourier results. The generalization to two-dimensional tokamak geometry produces some surprising results in the numerical computations[19] that have implemented this formalism. In particular, in Ref. 19 it is shown that the direct absorption of wave momentum (which is launched mostly in the radial direction) can produce *poloidal* plasma flows due to the effect of $k_{||}$ up-shifts[21,31] in the presence of a poloidal magnetic field. This direct wave momentum absorption can often dominate the bipolar flow-drive term that was the focus of previous investigations.

The main results of our paper are to be found in Eq. (56) for the particle flux, Eqs. (47) and (59) for the local energy absorption and heat flux, and Eqs. (53) – (55) and (67) – (71) for the total force. The total force is to be used in the macroscopic plasma momentum equation, Eq. (60). It was shown that this force can be separated into pieces that arise from dissipative and reactive interactions. The reactive portion reduces to the conventional cold-fluid ponderomotive force in an appropriate limit. The flux-surface-averaged poloidal and toroidal flows can be calculated from a subset of the nonlinear force terms, namely the dissipative terms given by Eq. (87). These are to be employed in calculating the flux surface averages which drive flows in neoclassical theory, viz. $\langle BF_{||}\rangle_{\psi}$ and $\langle RF_{\zeta}\rangle_{\psi}$. Flux surface averaged flows are found to be driven by two processes: the absorption of wave momentum and by dissipative stresses. The latter stresses are a generalization of the Reynolds stress that is prominent in the theory of sheared flows driven by turbulence. From the flows, the radial force, and the changes to the pressure gradient that result from rf heating, Eq. (88) gives the radial electric field.

The results of present calculations, as implemented in the AORSA rf code, complete a significant step in the development of tools that can lead to a physics understanding and a modeling capability for rf flow-drive experiments. The results of this rf modeling can be coupled with appropriate models for obtaining the flows from these forces, and then the effect of these flows on turbulence and transport can be computed.

The use of rf as a means of externally controlling sheared plasma flows could enable a deeper understanding of turbulence and transport barrier formation. Whereas turbulence-generated flows modify the waves that create the flows forming a "closed loop" system, rf-generated flows are "open loop" in that the waves can be manipulated externally and the plasma responses studied in a controlled context. Progress in this area will contribute to our understanding of wave-driven flows and possibly lead to useful practical applications for the control of fusion plasmas.

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Appendix A: Explicit kinetic representations of W, Q_W and Π_W

Explicit representations for the distribution functions, moments and the W operator are readily obtained in the Fourier representation where we expand

$$\mathbf{a} = \sum_k \mathbf{a}_k e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_k \mathbf{a}_k e^{i\mathbf{k}\cdot\mathbf{R} - i\delta_k} \tag{A1}$$

with

$$\delta_k = \frac{1}{\Omega}\mathbf{k}\cdot\mathbf{v}\times\mathbf{b}\,. \tag{A2}$$

In the analysis that follows, we will assume that the equilibrium magnetic field lines are sufficiently straight that we may use a constant B-field approximation for particle orbits.

First, as a basis for developing guiding-center expansions for second-order quantities such as $\langle\mathbf{J}\cdot\mathbf{E}\rangle$, we adopt a different (from customary) form for the first order distribution function (which will be available in guiding center variables)

$$f^{(1)} = \sum_k f_k e^{i\mathbf{k}\cdot\mathbf{R}} \tag{A3}$$

so that all the gyrophase dependence is explicitly in the f_k. Note that the $f_k = f_k(\mathbf{R})$ depend on the constants of motion (e.g. $\mathbf{R}$) through the equilibrium distribution function.

With these definitions, the linearized (first-order in rf electric field) Vlasov equation for f_k to leading order in ρ/L becomes

$$-i(\omega - k_{\|} v_{\|}) f_k - \Omega \frac{\partial f_k}{\partial \phi} = \frac{2 f_M}{\alpha^2} e^{-i\delta_k} \mathbf{a}_k \cdot \mathbf{v} \,, \tag{A4}$$

where f_M is the equilibrium distribution function, here taken as a Maxwellian ~ $\exp(-v^2/\alpha^2)$ with $\alpha^2 = 2T/m$. The diamagnetic rf currents are not being considered as we are interested in waves in ion cyclotron frequency range. We conjecture that the formalism can include both non-Maxwellian equilibrium and drift wave phenomena by using appropriate expressions for W in Eqs. (A20), (A22), (A24), and (A25). A step in this direction is indicated at the close of this appendix.

Performing the characteristic integrals in guiding center variables, we obtain

$$f_{k'} = -\frac{2 f_M}{\alpha^2 \Omega} e^{-i\omega_{k'}\phi/\Omega} \int_{-\infty}^{\phi} d\phi' \mathbf{a}_{k'} \cdot \mathbf{v}(\phi') e^{-i\delta_{k'}(\phi') + i\omega_{k'}\phi'/\Omega} \,, \tag{A5}$$

where we have changed dummy wavevector index from k to k′ for later convenience and $\omega_k = \omega - k_{\|} v_{\|}$. The usual expressions for the linear current perturbation can be obtained from using the Bessel representation of the trajectory integral and the definition

$$\mathbf{J} = Ze \int d^3v \sum_{k'} e^{i\mathbf{k}'\cdot\mathbf{R}} \mathbf{v} f_{k'}(\mathbf{R}) \,. \tag{A6}$$

Similarly,

$$\langle \mathbf{J} \cdot \mathbf{E} \rangle = \frac{m}{4} \int d^3v \sum_{k,k'} \mathbf{v} \cdot \mathbf{a}_k^* \, e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{R}} f_{k'}(\mathbf{R}) e^{i\delta_k} + cc \,. \tag{A7}$$

In order to perform the integral $\int d^3v$ at constant $\mathbf{r}$, we expand

$$e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{R}} f_{k'}(\mathbf{R}) = e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}} f_{k'}(\mathbf{r}) + \frac{1}{\Omega} \mathbf{v} \times \mathbf{b} \cdot \nabla \left(e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}} f_{k'}(\mathbf{r}) \right), \tag{A8}$$

While the second term has some characteristics (and the ordering) of the plasma diamagnetic response, it must be retained because it is essential to having a system for which satisfactory moment equations can be developed. We can now express $\langle \mathbf{J}\cdot\mathbf{E} \rangle$ as

$$\langle \mathbf{J} \cdot \mathbf{E} \rangle = \frac{m}{4} \int d^3v \sum_{k,k'} \mathbf{v} \cdot \mathbf{a}_k^* \, e^{i\delta_k} \times \left[e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}} f_{k'}(\mathbf{r}) + \frac{1}{\Omega} \mathbf{v} \times \mathbf{b} \cdot \nabla \left(e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}} f_{k'}(\mathbf{r}) \right) \right] + cc \,. \tag{A9}$$

First we *define* the lowest order part of Eq. (A9) as

$$\dot{w} = \frac{m}{4} \sum_{k,k'} e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}} \int d^3v\, \mathbf{v} \cdot \mathbf{a}_k^* f_{k'} e^{i\delta_k} + cc\,. \quad \text{(A10)}$$

In the following we will first calculate $\dot{w}$ and then return to Eq. (A9) to obtain the correction terms required for the rest of <**J**·**E**>.

Note that for the calculation of $f_{k'}$ and $\dot{w}$, we can replace **a** by Ze**E**/m which is independent of velocity since the terms arising from $\mathbf{B}^{(1)}$ are annihilated by the dot product with **v**. In particular, this allows $\mathbf{a}_k$ to be removed from the definition of the tensor L defined by

$$L = \int_0^{2\pi} d\phi\, \mathbf{v}(\phi)\, e^{i\delta_k(\phi) - i\omega_{k'}\phi/\Omega} \int_{-\infty}^{\phi} d\phi'\, \mathbf{v}(\phi')\, e^{-i\delta_{k'}(\phi') + i\omega_{k'}\phi'/\Omega}, \quad \text{(A11)}$$

where $f_M = n_0 f_{M\|} f_{M\perp}$, $f_{M\|} = (1/\alpha\pi^{1/2}) \exp(-v_\|^2/\alpha^2)$ and $f_{M\perp} = (1/\alpha^2\pi) \exp(-v_\perp^2/\alpha^2)$ so that $\int dv_\| f_{M\|} = 1$ and $2\pi \int dv_\perp v_\perp f_{M\perp} = 1$. L is closely related to the W tensor defined in Ref. 6

$$W = -\frac{2n_0 Z^2 e^2}{\alpha^2 \Omega m} \int dv_\| f_{M\|} \int dv_\perp v_\perp f_{M\perp}\, L\,, \quad \text{(A12)}$$

where n_0 is the density.

We evaluate L using the representation of the identity tensor

$$I = \frac{\mathbf{k}_\perp \mathbf{k}_\perp}{k_\perp^2} + \frac{\mathbf{b}\times\mathbf{k}_\perp \mathbf{b}\times\mathbf{k}_\perp}{k_\perp^2} + \mathbf{bb} \quad \text{(A13)}$$

to rewrite **v** in Eq. (27) as

$$\mathbf{v} = \frac{\mathbf{k}_\perp v_\perp}{k_\perp} \cos(\phi - \theta) + \frac{\mathbf{b}\times\mathbf{k}\, v_\perp}{k_\perp} \sin(\phi - \theta) + \mathbf{b} v_\|\,, \quad \text{(A14)}$$

where

$$\mathbf{k}_\perp = k_\perp (\mathbf{e}_x \cos\theta + \mathbf{e}_y \sin\theta)\,. \quad \text{(A15)}$$

A similar expansion is used for $\mathbf{v}(\phi')$ in terms of $\mathbf{k}'$. After some algebra which involves performing the gyrophase integrals, we find that L is given by

$$L = \sum_n \frac{2\pi\Omega e^{in(\theta'-\theta)}}{i(\omega - k_\| v_\| - n\Omega)}\; \mathbf{H}(\mathbf{k})^*\; \mathbf{H}(\mathbf{k}')\,, \quad \text{(A16)}$$

$$\mathbf{H}(\mathbf{k}) = \hat{\mathbf{k}}_\perp v_\perp \frac{n}{a} J_n(a) + \mathbf{b}\times\hat{\mathbf{k}}_\perp v_\perp i J_n'(a) + \mathbf{b} v_\| J_n(a)\,. \quad \text{(A17)}$$

It is straightforward to show that the present result is simply related to a rotation of the W matrix in the ‘Stix” frame, $W^{(0)}$ [see Ref. 6].

$$W(\mathbf{k},\mathbf{k}') = U^t(\mathbf{k}) \cdot W^{(0)}(\mathbf{k},\mathbf{k}') \cdot U(\mathbf{k}'), \tag{A18}$$

$$U(\mathbf{k}) = \frac{1}{k_\perp}(\mathbf{e}_x \mathbf{k}_\perp + \mathbf{e}_y \mathbf{b}\times\mathbf{k}_\perp) + \mathbf{bb} = \begin{pmatrix} \hat{k}_x & \hat{k}_y & 0 \\ -\hat{k}_y & \hat{k}_x & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A19}$$

Note that the U’s that pre and post multiply W have different arguments. Additionally, $W^{(0)}$ as given in Ref. 6 (where $\mathbf{k} = k\mathbf{e}_x$) must be generalized to include the phase factor $\exp(in(\theta'-\theta))$ inside the n sums, as in Eq. (A16) [see also Ref. 5]. The derivation given here is similar to that presented by Smithe,[23] where further details may be found.

Now consider the second term in Eq. (A9). This term is first order in ρ/L and can be written as

$$\begin{aligned} -\nabla\cdot\mathbf{Q}_w &= -\nabla\cdot\frac{m}{4\Omega}\sum_{k,k'} e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}} \int d^3v\, \mathbf{b}\times\mathbf{v}\ \mathbf{v}\cdot\mathbf{a}_k^* f_{k'} e^{i\delta_k} + cc \\ &= -\nabla\cdot\frac{i}{4}\sum_{k,k'} e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}} \frac{\partial}{\partial\mathbf{k}_\perp}\mathbf{E}_k^* \cdot W(\mathbf{k},\mathbf{k}')\cdot\mathbf{E}_k + cc\ . \end{aligned} \tag{A20}$$

Combining Eqs. (A9), (A10) and (A20) now gives us the desired energy moment expression

$$\langle \mathbf{J}\cdot\mathbf{E}\rangle = \dot{w} - \nabla\cdot\mathbf{Q}_w, \tag{A21}$$

where

$$\dot{w} = \frac{1}{4}\sum_{k,k'} e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}} \mathbf{E}_k^* \cdot W \cdot \mathbf{E}_{k'} + cc, \tag{A22}$$

The calculation of $<\nabla\mathbf{E}\cdot\mathbf{D}>$ is similar, the only difference being that there is an extra ∇ acting on $\mathbf{E}^*$ and $\mathbf{a}^*$ in Eq. (A9). The result is an immediate generalization of Eqs. (A20) - (A22). Thus

$$\frac{1}{4\pi}(\nabla\mathbf{E})\cdot\mathbf{D} = \mathbf{F}_0 - \nabla\cdot\Pi_w, \tag{A23}$$

$$\mathbf{F}_0 = \frac{1}{4}\sum_{kk'} e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}} \frac{\mathbf{k}}{\omega}\mathbf{E}_k^* \cdot W(\mathbf{k},\mathbf{k}')\cdot\mathbf{E}_{k'} + cc, \tag{A24}$$

and

$$\Pi_w = \frac{m}{4\Omega}\sum_{kk'} e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}} \int d^3v\, e^{i\delta_k} f_{k'} \mathbf{b}\times\mathbf{v}\,\mathbf{v}\cdot\mathbf{a}_k^* \frac{\mathbf{k}}{\omega} + cc\,. \tag{A25}$$

So far, we have only considered the linear distribution function to leading order in ρ/L, i.e. $f^{(1,0)}$. There is an additional contribution to **J**, $\mathbf{Q}_w$ and Π_w in first order that arises from $f^{(1,1)}$, driven by the last term on the rhs of Eq. (29), viz.

$$f_{k'}^{(1,1)} = \frac{1}{\Omega^2} e^{-i\omega_{k'}\phi/\Omega} \int_{-\infty}^{\phi} d\phi' e^{-i\delta_{k'}(\phi')+i\omega_{k'}\phi'/\Omega}\, \mathbf{a}_{k'}\cdot\mathbf{b}\times\nabla f_M\,, \tag{A26}$$

$$\begin{aligned}\langle \mathbf{J}\cdot\mathbf{E}\rangle = -\nabla\cdot\Bigg(&\frac{m}{4\Omega^2}\sum_{k,k'} e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}}\int d^3v\,\mathbf{v}\cdot\mathbf{a}_k^* e^{i\delta_{k'}-i\omega_{k'}\phi/\Omega} \\ &\int_{-\infty}^{\phi} d\phi' e^{-i\delta_{k'}(\phi')+i\omega_{k'}\phi'/\Omega}\, \mathbf{b}\times\mathbf{a}_{k'} f_M\Bigg) + cc + \ldots\,,\end{aligned} \tag{A27}$$

where … represents the terms already computed from $f^{(1,0)}$ and the term in parenthesis in the preceding equation is the new drift contribution to $\mathbf{Q}_w$. This term (and its counterparts in the conductivity and in Π_w) is associated with drift-wave physics and is frequently not retained in ICRF calculations because the drift frequency is small compared with the wave frequency, $\omega_* << \omega$.

Finally, we end this appendix by correcting an error in a previous publication. The nonlocal correction Π_w needs to be added to the terms explicitly given in the mixed polarization case, Sect. IV A. of Ref. 7. This corrects the results of that subsection [i.e. Eqs. (38) – (40) of Ref. 7] and leads to the conclusion that there is no net force for the electrostatic example given therein when the ions are not dissipative. The remainder of Ref. 7 is unaffected by this error.

Appendix B: Nonlinear force manipulations

1. Symmetrization of F_0

The zero order term of the nonlinear Lorenz force $\mathbf{F}_L$ is given by

$$\mathbf{F}_0 = \frac{1}{4}\sum_{kk'} e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}} \frac{\mathbf{k}}{\omega}\mathbf{E}_k^*\cdot W(\mathbf{k},\mathbf{k}')\cdot\mathbf{E}_{k'} + cc\,. \tag{B1}$$

To simplify the notation, we temporarily absorb the phase factors into the **E** fields, and treat the summation as implicit. Thus we consider

$$4\omega \mathbf{F}_0 = \mathbf{k}\,\mathbf{E}_k^* \cdot \mathrm{W}(\mathbf{k},\mathbf{k}')\cdot \mathbf{E}_{k'} + \mathrm{cc}\,. \tag{B2}$$

The first step is to interchange the dummy summation variables k and k′ and write the cc term instead as the explicit one

$$4\omega \mathbf{F}_0 = \mathbf{k}'\mathbf{E}_{k'} \cdot \mathrm{W}^*(\mathbf{k}',\mathbf{k})\cdot \mathbf{E}_k^* + \mathrm{cc}\,. \tag{B3}$$

Transposing the matrix and dotting the **E** vectors from the other sides gives

$$4\omega \mathbf{F}_0 = \mathbf{k}'\mathbf{E}_k^* \cdot \mathrm{W}^{*t}(\mathbf{k}',\mathbf{k})\cdot \mathbf{E}_{k'} + \mathrm{cc}\,. \tag{B4}$$

Eqs. (B2) and (B4) may be combined once the symmetry properties of W are specified. We divide W into its Hermitian and anti-Hermitian parts W = H + A where

$$\mathrm{H}^{*t}(\mathbf{k}',\mathbf{k}) = \mathrm{H}(\mathbf{k},\mathbf{k}')\,, \tag{B5}$$

$$\mathrm{A}^{*t}(\mathbf{k}',\mathbf{k}) = -\mathrm{A}(\mathbf{k},\mathbf{k}')\,. \tag{B6}$$

Note that our Hermitian conjugate operation involves a complex conjugation, a transpose of the spatial elements of the tensor and an interchange of k and k′. Since W is like a conductivity, the dissipative physics is contained in H while the reactive physics is in A. These symmetries of the dissipative and reactive pieces can be verified by looking at the explicit result for W given in Eq. (16) of Ref. 6. [Note that transposed elements effectively conjugate the explicit imaginary components of W but not the Z-functions.] With these symmetries we can rewrite Eqs. (B2) and (B4) as

$$4\omega \mathbf{F}_0 = \mathbf{k}\,\mathbf{E}_k^* \cdot \mathrm{H}(\mathbf{k},\mathbf{k}')\cdot \mathbf{E}_{k'} + \mathbf{k}\,\mathbf{E}_k^* \cdot \mathrm{A}(\mathbf{k},\mathbf{k}')\cdot \mathbf{E}_{k'} + \mathrm{cc}\,, \tag{B7}$$

$$4\omega \mathbf{F}_0 = \mathbf{k}'\mathbf{E}_k^* \cdot \mathrm{H}(\mathbf{k},\mathbf{k}')\cdot \mathbf{E}_{k'} - \mathbf{k}'\mathbf{E}_k^* \cdot \mathrm{A}(\mathbf{k},\mathbf{k}')\cdot \mathbf{E}_{k'} + \mathrm{cc}\,. \tag{B8}$$

Taking 1/2 the sum of the above equations and reinserting the suppressed notation yields the desired result, Eqs. (53) – (55) of the main text.

2. Alternative form for Π_{DE}

We begin from the definition

$$\Pi_{DE} = \frac{1}{16\pi}\mathbf{D}\mathbf{E}^* + \mathrm{cc}\,. \tag{B9}$$

Manipulations can cast this into a form that involves similar quantities to those in Π_w. The *linearized* momentum equation crossed with **b** yields

$$-\frac{4\pi}{\Omega}\mathbf{b}\times\mathbf{J} + \frac{4\pi i Ze}{m\omega\Omega}\nabla\cdot \mathrm{P}\times\mathbf{b} + \mathbf{D}_\perp = -\frac{i\omega_p^2}{\omega\Omega}\mathbf{b}\times\mathbf{E}\,, \tag{B10}$$

which can be used to eliminate the perpendicular part of **D**. Here P is the linearized pressure tensor. This results in

$$\Pi_{DE} = \frac{1}{4\Omega}\mathbf{b}\times\mathbf{J}\mathbf{E}^* - \frac{iZe}{4m\omega\Omega}\nabla\cdot \mathrm{P}\times\mathbf{b}\mathbf{E}^* - \frac{i\omega_p^2}{16\pi\omega\Omega}\mathbf{b}\times\mathbf{E}\mathbf{E}^* + \frac{1}{16\pi}\mathbf{D}_{||}\mathbf{E}^* + \mathrm{cc} \quad .\text{(B11)}$$

The terms in this equation are cast into velocity integral form noting the following points. The lowest order expression may be employed for $\nabla\cdot\mathrm{P}$ *viz.* $\nabla\cdot\mathrm{P}\rightarrow i\mathbf{k}'\cdot\mathrm{P}$ since there is already a slow divergence in front of the entire Π_{DE} term. The distinction between $\exp(i\delta_{k'})$ and $\exp(i\delta_k)$ need not be retained to the order required. Species summed quasineutrality is used to eliminate some terms, e.g. the term on the rhs of Eq. (B10) vanishes under the assumption of species-summed quasineutrality. Collecting the results, we obtain

$$\Pi_{DE} = \frac{m}{4\Omega}\sum_{kk'} e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}}\int d^3v\, e^{i\delta_{k'}} f_{k'}\mathbf{b}\times\mathbf{v}\left(1-\frac{\mathbf{k}'\cdot\mathbf{v}}{\omega}\right)\frac{Ze}{m}\mathbf{E}_k^* + \frac{1}{16\pi}\mathbf{D}_{||}\mathbf{E}^* + \mathrm{cc}\,. \text{ (B12)}$$

Appendix C: Expressions for X_d and X_r in terms of W

To obtain an expression for X_d in terms of W, we begin by writing $\dot{w}$ in terms of $\mathbf{H}(\mathbf{k})$ as in Appendix A [see Eqs. (A16) and (A17)],

$$\dot{w} = -\frac{Z^2e^2}{2m\alpha^2}\int d^3v\, f_M \sum_n \frac{e^{in(\theta'-\theta)}}{i(\omega - k_{||}v_{||} - n\Omega)}\; \mathbf{E}_k^*\cdot\mathbf{H}(\mathbf{k})^*\; \mathbf{H}(\mathbf{k}')\cdot\mathbf{E}_{k'} + \mathrm{cc}\,, \qquad \text{(C1)}$$

where the implicit sums on **k** and **k′** and the phase factors $\exp[i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}]$ are absorbed into the fields. Comparing Eqs. (47) and (69) is evident that we can use the expression for $\dot{w}$ to obtain $\dot{w}_\perp$ by making the replacement $\mathbf{v}\cdot\mathbf{a}\rightarrow\mathbf{v}_\perp\cdot\mathbf{a}_\perp$, or equivalently

$$\frac{Ze}{m}\mathbf{v}\cdot\mathbf{E}^* \rightarrow \frac{Ze}{m}\mathbf{v}_\perp\cdot\left[\mathbf{E}_\perp^*\left(1-\frac{k_{||}v_{||}}{\omega}\right)+\frac{v_{||}}{\omega}\mathbf{k}_\perp E_{||}^*\right]. \qquad \text{(C2)}$$

Thus, from Eq. (C1)

$$\dot{w}_\perp = -\frac{Z^2e^2}{2m\alpha^2}\int d^3v\, f_M\sum_n\frac{e^{in(\theta'-\theta)}}{i(\omega-k_{||}v_{||}-n\Omega)}$$
$$\times\left[\mathbf{E}_{k\perp}^*\left(1-\frac{k_{||}v_{||}}{\omega}\right)+\frac{v_{||}}{\omega}\mathbf{k}_\perp E_{k||}^*\right]\cdot\mathbf{H}(\mathbf{k})^*\;\mathbf{H}(\mathbf{k}')\cdot\mathbf{E}_{k'} + \mathrm{cc}. \qquad \text{(C3)}$$

Because $\dot{w}_\perp$ is a dissipative term, and only the resonant particles will contribute, we may set $\omega - k_{||}v_{||} = n\Omega$ in the $(1 - k_{||}v_{||}/\omega)$ factor. (Alternatively, we may add and subtract $n\Omega$

and observe that the $\omega - k_{\|}v_{\|} - n\Omega$ cancels the resonant denominator. The velocity integrals now evaluate to zero, using parity in $v_{\|}$ and taking + cc into account.) Furthermore, from the explicit expression for **H** given in Eq. (A17)

$$\mathbf{H}\cdot\mathbf{k}_{\perp} = n\Omega J_n(a)\,, \qquad \text{(C4)}$$

$$H_{\|} = v_{\|}J_n(a)\,, \qquad \text{(C5)}$$

so

$$\dot{w}_{\perp} = -\frac{Z^2e^2}{2m\alpha^2}\int d^3v\, f_M \sum_n \frac{e^{in(\theta'-\theta)}}{i(\omega - k_{\|}v_{\|} - n\Omega)}\,\frac{n\Omega}{\omega}\,\mathbf{E}_k^*\cdot\mathbf{H}(\mathbf{k})^*\;\mathbf{H}(\mathbf{k}')\cdot\mathbf{E}_{k'} + cc\,. \qquad \text{(C6)}$$

For the W tensor embedded in $\dot{w}$, we can separate out the individual gyro-harmonic contributions by defining

$$W = \sum_n W_n\,, \qquad \text{(C7)}$$

so that

$$\dot{w} = \frac{1}{4}\sum_n \mathbf{E}_k^*\cdot W_n\,(\mathbf{k},\mathbf{k}')\cdot\mathbf{E}_{k'} + cc\,, \qquad \text{(C8)}$$

$$\dot{w}_{\perp} = \frac{1}{4}\sum_n \frac{n\Omega}{\omega}\mathbf{E}_k^*\cdot W_n\,(\mathbf{k},\mathbf{k}')\cdot\mathbf{E}_{k'} + cc\,, \qquad \text{(C9)}$$

and

$$X_d = \frac{1}{8\omega}\sum_n n\mathbf{E}_k^*\cdot W_n\,(\mathbf{k},\mathbf{k}')\cdot\mathbf{E}_{k'} + cc\,. \qquad \text{(C10)}$$

This compact form for X_d has been noted previously in the 1-D case considered in Ref. 6.

To obtain an expression for X_r we begin with Eq. (68) in the present notation

$$X_r = \frac{m}{8\Omega}\int d^3v\;\mathbf{b}\cdot\mathbf{v}\times\mathbf{a}_k^*\;f_{k'}e^{i\delta_k} + cc\,, \qquad \text{(C11)}$$

where δ_k and $\delta_{k'}$ are indistinguishable to lowest significant order. Making use of the fact that $\partial/\partial\mathbf{k}$ brings down $(i/\Omega)\mathbf{v}\times\mathbf{b}$ when operating on $\exp(i\delta_k)$, pulling $\partial/\partial\mathbf{k}$ outside the integral and correcting for

$$\frac{\partial}{\partial\mathbf{k}}\cdot\mathbf{a}_k = \frac{1}{\omega}\mathbf{v}\cdot\mathbf{a}_k \qquad \text{(C12)}$$

yields

$$X_r = \frac{im}{8}\frac{\partial}{\partial\mathbf{k}}\cdot\int d^3v\;\mathbf{a}_{k\perp}^*\;f_{k'}e^{i\delta_k} - \frac{im}{8\omega}\int d^3v\;\mathbf{v}\cdot\mathbf{a}_k^*\;f_{k'}e^{i\delta_k} + cc\,. \qquad \text{(C13)}$$

The first term in the preceding equation is the k-divergence of the lowest order nonlinear Lorentz force $\mathbf{F}_0$, while the second term is proportional to $\dot{w}$. Employing Eqs. (47) and (50) yields the final result

$$\begin{aligned} X_r &= \frac{i}{8\omega}\frac{\partial}{\partial \mathbf{k}_\perp}\cdot\left(\mathbf{k}\mathbf{E}_k^*\cdot W(\mathbf{k},\mathbf{k}')\cdot\mathbf{E}_{k'}\right) - \frac{i}{8\omega}\mathbf{E}_k^*\cdot W(\mathbf{k},\mathbf{k}')\cdot\mathbf{E}_{k'} + cc \\ &= \frac{i}{8\omega}\left[\mathbf{E}_k^*\cdot W(\mathbf{k},\mathbf{k}')\cdot\mathbf{E}_{k'} + \mathbf{k}_\perp\cdot\frac{\partial}{\partial \mathbf{k}_\perp}[\mathbf{E}_k^*\cdot W(\mathbf{k},\mathbf{k}')\cdot\mathbf{E}_{k'}]\right] + cc\,. \end{aligned} \tag{C14}$$

Appendix D: Equivalent forms of the cold-fluid ponderomotive force

In this appendix we outline the proof of the equivalence of Eqs. (72) and (82) in general magnetic geometry. For notational convenience in this appendix, the symbol ψ instead of ψ_p will be used for the ponderomotive potential since there is no possibility of confusion with the flux function. The proof begins by comparing the first terms in each of these forms. They differ by a gradient of the susceptibility tensor, viz.

$$-n\nabla\psi + n\partial\psi = \frac{1}{16\pi}\left[(\nabla\mathbf{E}^*)\cdot\mathbf{D})\right] + cc\,, \tag{D1}$$

where the operator ∂ is a gradient holding $\mathbf{E}$ and $\mathbf{E}^*$ fixed, i.e. it operates only on χ.

Next consider the divergence term in Eq. (82). We use $\mathbf{D} = 4\pi i\mathbf{J}/\omega$ to express $\mathbf{D}$ in terms of $\mathbf{J} = Zen\mathbf{u}$ and then Eq. (75) to express $\mathbf{E}$ in terms of $\mathbf{u}$ to obtain

$$\mathbf{D}\mathbf{E}^* + 4\pi nm\mathbf{u}\mathbf{u}^* = -4\pi inm\frac{\Omega}{\omega}\,\mathbf{u}\;\mathbf{u}^*\times\mathbf{b}\,. \tag{D2}$$

For the tensor on the rhs, we take the cc explicitly

$$\begin{aligned} \mathbf{D}\mathbf{E}^* + 4\pi nm\mathbf{u}\mathbf{u}^* + cc &= -4\pi inm\frac{\Omega}{\omega}\left(\frac{1}{2}I_\perp\mathbf{b}\cdot\mathbf{u}\times\mathbf{u}^* + \mathbf{b}u_{\|}\mathbf{u}^*\times\mathbf{b}\right) + cc \\ &= -16\pi\left(I_\perp BM_{\|} - \mathbf{B}\mathbf{M}_\perp\right). \end{aligned} \tag{D3}$$

Employing the identities $\nabla\cdot(\mathbf{B}\mathbf{M}_\perp) = B\nabla_{\|}\mathbf{M}_\perp$ and $\nabla\cdot(I_\perp\phi) = \nabla_\perp\phi - \kappa\phi + \mathbf{b}\phi\nabla_{\|}\ln B$ where $\kappa = \mathbf{b}\cdot\nabla\mathbf{b}$ and ϕ is any scalar, we obtain

$$\nabla\cdot\left(I_\perp BM_{\|} - \mathbf{B}\mathbf{M}_\perp\right) = \nabla_\perp(BM_{\|}) - \kappa BM_{\|} + \mathbf{b}M_{\|}\nabla_{\|}B - B\nabla_{\|}\mathbf{M}_\perp\,. \tag{D4}$$

Next, we turn to a manipulation of $n\partial\psi$. By operating with ∂ on Eq. (75), and working in a mixed Cartesian tensor dyad notation one obtains

$$-i\omega\partial_i\mathbf{u} - \Omega\partial_i\mathbf{u}\times\mathbf{b} = \nabla_i\Omega\mathbf{u}\times\mathbf{b} + \Omega\mathbf{u}\times\nabla_i\mathbf{b}\,. \tag{D5}$$

Noting that the lhs of the Eqs. (D5) and (75) are the same, and defining the operator C by $C\mathbf{u} = \mathbf{a}$ allows Eq. (D5) to be written as

$$\partial_i \mathbf{u} = \nabla_i \Omega C^{-1} \mathbf{u} \times \mathbf{b} + \Omega C^{-1} \mathbf{u} \times \nabla_i \mathbf{b} . \tag{D6}$$

Then, dotting the previous equation with $\mathbf{E}^*$ and using the fact the C is anti-Hermitian results in

$$\partial_i \mathbf{u} \cdot \mathbf{E}^* = -\frac{m}{Ze} \nabla_i \Omega \mathbf{u}^* \cdot \mathbf{u} \times \mathbf{b} - \frac{m}{Ze} \Omega \mathbf{u}^* \cdot \mathbf{u} \times \nabla_i \mathbf{b} . \tag{D7}$$

The first term in Eq. (D7) can be immediately related to $M_{||}$. In the second term we can interchange dot and cross to obtain an expression in terms of $\mathbf{M} \propto \mathbf{u}^* \times \mathbf{u}$. Inserting the other constants required to form ψ we obtain

$$-n\partial\psi = M_{||} \nabla B + B(\nabla \mathbf{b}) \cdot \mathbf{M} . \tag{D8}$$

Finally, working with the $\nabla \times \mathbf{M}$ term on the lhs of Eq. (72), we can write

$$\mathbf{B} \times \nabla \times \mathbf{M} = \nabla(BM_{||}) - (\nabla \mathbf{B}) \cdot \mathbf{M} - B\nabla_{||}\mathbf{M} , \tag{D9}$$

where we have used $\mathbf{B} \times \nabla \times \mathbf{M} = (\nabla \mathbf{M}) \cdot \mathbf{B} - \mathbf{B} \cdot \nabla \mathbf{M}$ and $\nabla(\mathbf{M} \cdot \mathbf{B}) = (\nabla \mathbf{M}) \cdot \mathbf{B} + (\nabla \mathbf{B}) \cdot \mathbf{M}$. Combining Eqs. (D8) and (D9) gives

$$-n\partial\psi + \mathbf{B} \times \nabla \times \mathbf{M} = \nabla(BM_{||}) - B\nabla_{||}\mathbf{M} . \tag{D10}$$

The proof of equivalence of the two forms of the ponderomotive force requires showing that the rhs of Eq. (D10) equals the rhs of Eq. (D4). This follows immediately on noting that $B\nabla_{||}\mathbf{M} = B\nabla_{||}(\mathbf{M}_\perp + \mathbf{b}M_{||}) = B\nabla_{||}\mathbf{M}_\perp + \kappa BM_{||} + \mathbf{B}\nabla_{||}M_{||}$.

Appendix E: Flux surface averages and identities

Working in orthogonal (ψ, θ, ζ) coordinates and considering axisymmetric magnetic geometry we have

$$\nabla = \frac{\mathbf{e}_\psi}{h_\psi} \frac{\partial}{\partial \psi} + \frac{\mathbf{e}_\theta}{h_\theta} \frac{\partial}{\partial \theta} + \frac{\mathbf{e}_\zeta}{h_\zeta} \frac{\partial}{\partial \zeta} , \tag{E1}$$

$$\begin{aligned} h_\psi &= \frac{1}{RB_\theta}, \\ h_\theta &= JB_\theta, \\ h_\zeta &= R, \end{aligned} \tag{E2}$$

where the definition of the θ coordinate is unspecified, and determines the Jacobian

$$J = h_\psi h_\theta h_\zeta = \frac{\partial(x, y, z)}{\partial(\psi, \theta, \zeta)} \,. \tag{E3}$$

Then the flux surface average of any quantity Q is defined as

$$\langle Q \rangle_\psi = \frac{1}{\upsilon} \int \frac{d\zeta}{2\pi} \int d\theta\, JQ \equiv \int \frac{ds}{B} Q \,, \tag{E4}$$

$$\upsilon = \int d\theta\, J \equiv \int \frac{ds}{B} \,, \tag{E5}$$

where ds is arc length along the total B-field, and the last form assumes coverage of the magnetic surface.

In the following identities, Q, **A** and Π are an arbitrary scalar, vector, and tensor:

$$\langle \nabla \cdot \mathbf{A} \rangle_\psi = \frac{1}{\upsilon} \frac{\partial}{\partial \psi} \upsilon \langle R B_\theta A_\psi \rangle_\psi = \frac{1}{\upsilon} \frac{\partial}{\partial \psi} \upsilon \langle \nabla \psi \cdot \mathbf{A} \rangle_\psi \,, \tag{E6}$$

$$\langle B \nabla_{||} Q \rangle_\psi = 0 \,, \tag{E7}$$

where $\mathbf{b} \cdot \nabla = \nabla_{||} = \partial/\partial s$. For symmetric Π it can be shown that

$$\Pi : \nabla(R \mathbf{e}_\zeta) = 0 \tag{E8}$$

so it follows that for symmetric Π

$$\langle R \mathbf{e}_\zeta \cdot \nabla \cdot \Pi \rangle_\psi = \langle \nabla \cdot \Pi \cdot R \mathbf{e}_\zeta \rangle_\psi = \frac{1}{\upsilon} \frac{\partial}{\partial \psi} \upsilon \langle R^2 B_\theta \Pi_{\psi\zeta} \rangle_\psi \,. \tag{E9}$$

For a CGL tensor $\Pi_{cgl} = S\, \mathsf{I} + Q\mathbf{bb}$ (where S and Q are scalars),

$$\langle \mathbf{B} \cdot \nabla \cdot \Pi_{cgl} \rangle_\psi = \langle \mathbf{B} \cdot \nabla \cdot (Q\mathbf{bb}) \rangle_\psi = -\langle Q \nabla_{||} B \rangle_\psi \,, \tag{E10}$$

where we have used the following identity:

$$\nabla \cdot (Q\mathbf{bb}) = Q\kappa + \mathbf{B} \nabla_{||}(Q/B) \,, \tag{E11}$$

with $\boldsymbol{\kappa} = \nabla_{||}\mathbf{b}$ the magnetic curvature and $\boldsymbol{\kappa} \cdot \mathbf{b} = 0$.

References

[1] H. Biglari, P.H. Diamond and P.W. Terry, Phys. Fluids B **2**, 1 (1990); K.H. Burrell et al., Plasma Phys. Controlled Fusion **34**, 1859 (1992); P.W. Terry, Rev. Mod. Phys. **72**, 109 (2000).
[2] G.G. Craddock and P.H. Diamond, Phys. Rev. Lett. **67**, 1535 (1991).
[3] G.G. Craddock, P.H. Diamond, M. Ono and H. Biglari, Phys. Plasmas **1**, 1944 (1994).
[4] L.A. Berry, E.F. Jaeger and D.B. Batchelor, Phys. Rev. Lett. **82**, 1871 (1999).
[5] E.F. Jaeger, L.A. Berry and D.B. Batchelor, Phys. Plasmas **7**, 641 (2000).
[6] E. F. Jaeger, L. A. Berry, and D. B. Batchelor Phys. Plasmas **7**, 3319 (2000).
[7] J.R. Myra and D.A. D'Ippolito, Phys. Plasmas **7**, 3600 (2000).
[8] A. G. Elfimov, G. Amarante Segundo, R. M. O. Galvão, and I. C. Nascimento Phys. Rev. Lett. **84**, 1200 (2000).
[9] M. Ono, P. Beiersdorfer, R. Bell, S. Bernabei, *et al.*, Phys. Rev. Lett. **60**, 294 (1988).
[10] T. Seki, K. Kawahata, M. Ono, K. Ida, *et al.*, in *AIP Conference Proceedings 244 – Radio Frequency Power in Plasmas*, Charleston, SC (American Institute of Physics, New York, 1991), p. 138.
[11] J. D. Moody, M. Porkolab, C. L. Fiore, F. S. McDermott, Y. Takase, J. Terry, and S. M. Wolfe, Phys. Rev. Lett. **60**, 298 (1988).
[12] B. LeBlanc, S. Batha, R. Bell, S. Bernabei, *et al*. Phys. Plasmas **2**, 741 (1995).
[13] J.R. Wilson, R.E. Bell, S. Bernabei, K. Hill, et al., Phys. Plasmas **5**, 1721 (1998).
[14] B.P. LeBlanc, R.E. Bell, S. Bernabei, J.C. Hosea, *et al.*, Phys. Rev. Lett. **82**, 331 (1999).
[15] C.K. Phillips, M.G. Bell, R.E. Bell, S. Bernabei, et al., Nucl. Fusion **40**, 461 (2000).
[16] R. Cesario, A. Cardinali, C. Castaldo, M. Leigheb, *et al.*, Phys. Plasmas **8**, 4721 (2001).
[17] C. Castaldo et al., in Proceedings of the 19th IAEA Fusion Energy Conference, Lyon, France, 2002 (IAEA, Vienna, 2002).
[18] E.F. Jaeger, L.A. Berry, E.D. D'Azevedo, D.B. Batchelor and M.D. Carter, Phys. Plasmas **8**, 1573 (2001).
[19] E.F. Jaeger, L.A. Berry, J.R. Myra, D.B. Batchelor, et al., Phys. Rev. Lett. **90**, 195001 (2003).
[20] J. Wright and P. Bonoli, "Modeling Mode Conversion in Alcator-CMOD and Asdex-Upgrade with a Converged Spectral Code", report in progress, (2003).
[21] F.W. Perkins, Nucl. Fusion **17**, 1197 (1977).
[22] E. Nelson-Melby, M. Porkolab, P.T. Bonoli, Y. Lin, A. Mazurenko and S.J. Wukitch, Phys. Rev. Lett. **90**, 155004 (2003); Y. Lin, S.J. Wukitch, P.T. Bonoli, E. Marmar et al., Plasma Phys. Controlled Fusion **45**, 1013 (2003).
[23] D.N. Smithe, Plasma Phys. Controlled Fusion **31**, 1105 (1989).
[24] M. Brambilla, *Kinetic Theory of Plasma Waves*, (Clarendon Press, Oxford, 1998).
[25] B.D. McVey, R.S. Sund, and J.E. Scharer, Phys. Rev. Lett. **55**, 507 (1985).
[26] N.C. Lee and G.K. Parks, Phys. Fluids **26**, 724 (1983); and refs. therein.
[27] P.J. Catto and J.R. Myra, Phys. Fluids B **1**, 1193 (1989) ; and refs. therein.
[28] J.R. Myra and D.A. D'Ippolito, Phys. Plasmas **9**, 3867 (2002); and refs. therein.
[29] S.P. Hirshman, Nucl. Fusion **18**, 917 (1978).
[30] S.P. Hirshman, Phys. Fluids **31**, 3150 (1988).
[31] A.K. Ram and A. Bers, Phys. Fluids B **3**, 1059 (1991).