On relaxing the Boussinesq approximation in scrape-off layer turbulence (SOLT) model simulations*

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Abstract

In the Boussinesq approximation, spatial variations in the plasma density are ignored in the convection of vorticity, leading to an equation of evolution for $n\nabla^2\phi$ rather than $\nabla\cdot(n\nabla\phi)$, where n and ϕ are the density and potential. In the blob-dominated turbulence of the near edge and SOL, density and potential fluctuation scales are similar, making this approximation hard to justify. The shortcomings of the approximation have been shown in studies of isolated blob motion [1], while recent studies of SOL turbulence suggest a relatively weak effect [2]. The numerical hardships and physical advantages of relaxing the approximation in the SOLT model [3] are discussed. On the algorithmic side, a Poisson solve for the potential becomes $n\nabla^2\phi+\nabla n\cdot\nabla\phi+\rho=0$, to be solved for ϕ at each time step, given the evolved *turbulent* fields n and ρ. We present multi-grid relaxation and direct (sparse matrix) methods for doing so. Eliminating the approximation allows us to add physics to the SOLT model that could not otherwise be included, such as *self-consistent* ion diamagnetic drift evolution.

[1] G. Yu et al., Phys. Plasmas **13**, 042508 (2006).

[2] K. Bodi et al., 38th EPS Conf. Plasma Phys. (2011).

[3] D.A. Russell et al., Phys. Plasmas **16**, 122304 (2009).

Outline

I. Model Equations
- the Boussinesq approximation

II. Numerical Method
- Multigrid (MG)

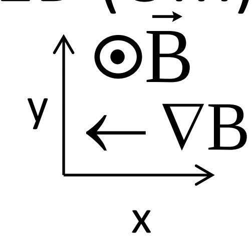
III. Blob dynamics
- Boussinesq blobs are fragile
- MG: the agony and the ecstasy

IV. Ion pressure effects
- Enhanced blob polarization
- A blob’s radial E-field (mean flow)
- At the edge

V. Summary

SOLT model equations (reduced Braginskii)

2D (OM)



Vorticity

Bohm units

General Case

$$\nabla\cdot(n\nabla\phi)+\rho+\nabla^2 P_i = 0 :$$

$$\frac{d}{dt}\rho = \frac{2\rho_s}{R}\partial_y P - J_{//} + \mu\nabla^2\rho +$$

$$+\tfrac{1}{2}\nabla^2(v_E\cdot\nabla P_i) - \tfrac{1}{2}v_E\cdot\nabla(\nabla^2 P_i) +$$

$$-\hat{b}\cdot\nabla n\times\nabla v_E^2 - \nabla\cdot\left(\frac{dn}{dt}\nabla\phi\right)$$

+ ... other gyro-viscosity terms, here ignored.

Boussinesq Approximation

$$\nabla^2\phi+\rho_B+\nabla^2 P_i / n = 0 :$$

$$\frac{d}{dt}\rho_B = \frac{2\rho_s}{R}\frac{1}{n}\partial_y P - J_{//}/n + \mu\nabla^2\rho_B$$

$$+\frac{1}{n}\left[\tfrac{1}{2}\nabla^2(v_E\cdot\nabla P_i) - \tfrac{1}{2}v_E\cdot\nabla(\nabla^2 P_i)\right]$$

Widely used

<u>Difficult to justify</u>

All fields are turbulent: n = n(x,y,t), etc.
We do not expand about ambient profiles.
Self-consistent O(1) fluctuations are supported.

SOLT model equations
(continues)

Density (quasi-neutral)

$$\frac{d}{dt}n = J_{//,n} + D_n \nabla^2 n + S_n$$

Electron Temperature

$$\frac{d}{dt}T_e = q_{//,e}/n + D_{Te}\nabla^2 T_e + S_{Te}$$

Ion Temperature

$$\frac{d}{dt}T_i = q_{//,i}/n + D_{Ti}\nabla^2 T_i + S_{Ti}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_E \cdot \nabla, \quad v_E = \hat{b} \times \nabla\phi \quad (\hat{b} \cdot \nabla \times v_E = \nabla^2 \phi), \nabla = \nabla_\perp$$

$J_{//}$ models electron drift waves on the closed field lines and sheath physics, through closure relations, in the SOL.
$q_{//}$ models heat flux in the SOL.

See J.R. Myra et al., Phys. Plasmas ***18****, 012305 (2011)*
for details of $J_{//}$ *and* $q_{//}$.

Numerical Method

The Problem

Extract ϕ from

$$\nabla \cdot (n\nabla\phi) + \rho + \nabla^2 P_i = 0$$

An exact linear solve grows prohibitive on larger grids.

Boussinesq $\Rightarrow$ Poisson:

$$\nabla^2\phi + \rho_B + \nabla^2 P_i / n = 0$$

This suggests a relaxation method for the general case:

$$n\nabla^2\phi^{m+1} + \nabla n \cdot \nabla\phi^m + \rho + \nabla^2 P_i = 0$$

It's simple - a straightforward application of the Poisson solver.

But, it's slow to converge at long scales where the mean flow lives!

And, there is an instability lurking at long scales for smooth density profiles.

von Neumann multiplier

$$\xi(k) = \frac{\nabla n \cdot k}{n\, k^2} \sim \frac{1}{L_n k}$$

...a result that assumes n is "sufficiently smooth."
But this is not the case in edge turbulence: δn and $\delta\phi$ have comparable and broadly distributed scales,
and instability *may* not be observed in realistic test cases.

This looks like a job for **Multigrid**.

Numerical Method
Multigrid (MG)

Short-cycle Poisson relaxation on nested coarse grids

(0) Relax ϕ, from an initial guess, ν_0 times on the simulation grid (N_x, N_y):

$$\nabla^2\varphi^{(m)} + \nabla n \cdot \nabla\varphi^{(m-1)}/n = -\rho/n\ ;\ m = 1,\ldots,\nu_0$$

$\varphi^{(0)} =$ a good guess, e.g., the previous result from simulation.

error $\varepsilon = \phi_{\text{Exact}} - \phi^{(\nu_0)}$

With relaxation, the error is reduced by numerical diffusion ($\sim k^p$) so long scale-length errors persist. The defect (D) drives the error:

$$D \equiv \nabla^2\varphi^{(\nu_0)} + \nabla n \cdot \nabla\varphi^{(\nu_0)}/n + \rho/n$$

$$\nabla^2\varepsilon + \nabla n \cdot \nabla\varepsilon/n + \rho/n = -D$$

Solve this, correct $\phi^{(\nu_0)}$, and we're done. But it's the *same problem* again, and we want to reduce the error at long scales, in particular. To do so efficiently, project the defect onto a 4x coarser grid, ½ (N_x, N_y), and

(1) Relax the error there:

$$\nabla^2\varepsilon_1^{(m)} + \nabla n_1 \cdot \nabla\varepsilon_1^{(m-1)}/n_1 + \rho_1/n_1 = -P(D)\ ;\ \varepsilon_1^{(0)} = 0\ ;\ m = 1,\ldots,\nu_1$$

where $(n_1, \rho_1) = P(n, \rho)$, n and ρ projected onto the coarse grid.

Multi-Grid (MG)
(continues)

Don't stop yet! There are errors in the error, especially at long scales…

$$D_1 \equiv \nabla^2 \varepsilon_1^{(\nu_1)} + \nabla n_1 \cdot \nabla \varepsilon_1^{(\nu_1)} / n_1 + \rho_1 / n_1$$

Project the defect onto a 4x coarser grid, ¼ (N_x, N_y), and

(2) Relax the error there:

$$\nabla^2 \varepsilon_2^{(m)} + \nabla n_2 \cdot \nabla \varepsilon_2^{(m-1)} / n_2 + \rho_2 / n_2 = -P(D_1)\ ;\ \varepsilon_2^{(0)} = 0\ ;\ m = 1,\ldots,\nu_2$$

where $(n_2, \rho_2) = P(n_1, \rho_1)$, the projection of the fields onto a still coarser grid.

A pattern emerges!
Continue projection/relaxation down through N_C coarse grids until the problem is exactly solved, *easily*, on the $(N_X, N_Y)/2^{N_C}$ grid.

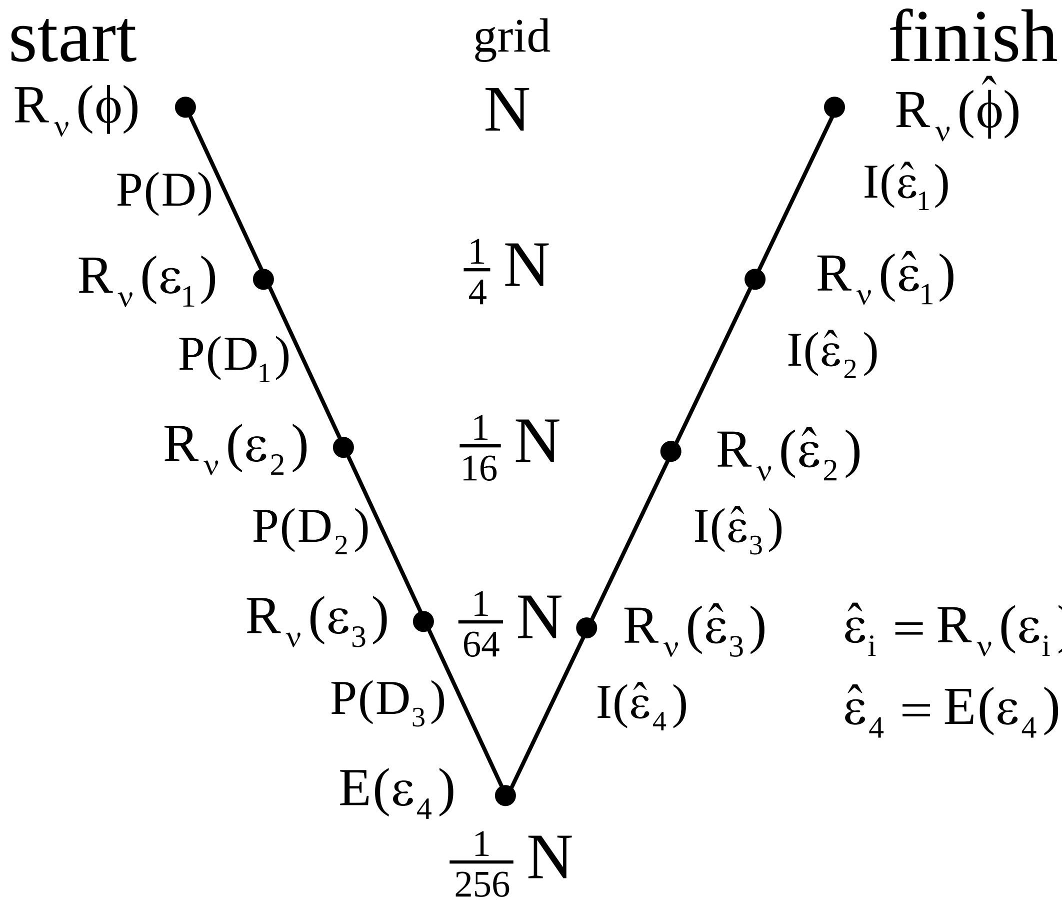
Return to the original, finest grid through the coarse grids by
(a) Interpolating the error to the next finer grid, and correcting the relaxed error there and
(b) Relaxing the corrected error (or ϕ) to remove high-k errors from the interpolation.
Continue "up" until ϕ is corrected on the simulation (finest) grid.

This basic MG algorithm is the V-cycle.

Reference: W. L. Briggs, V. E. Henson and S. F. McCormick, A Multigrid Tutorial, 2nd ed., SIAM (2000) .

Multi-Grid (MG)
(continues)

V-cycle
4 coarse grids



$$\hat{\phi} = R_\nu(\phi) + I(\hat{\varepsilon}_1)$$

$$\hat{\varepsilon}_i = R_\nu(\varepsilon_i) + I(\hat{\varepsilon}_{i+1})$$

$$\hat{\varepsilon}_4 = E(\varepsilon_4)$$

R_ν : relax ν times

P : project

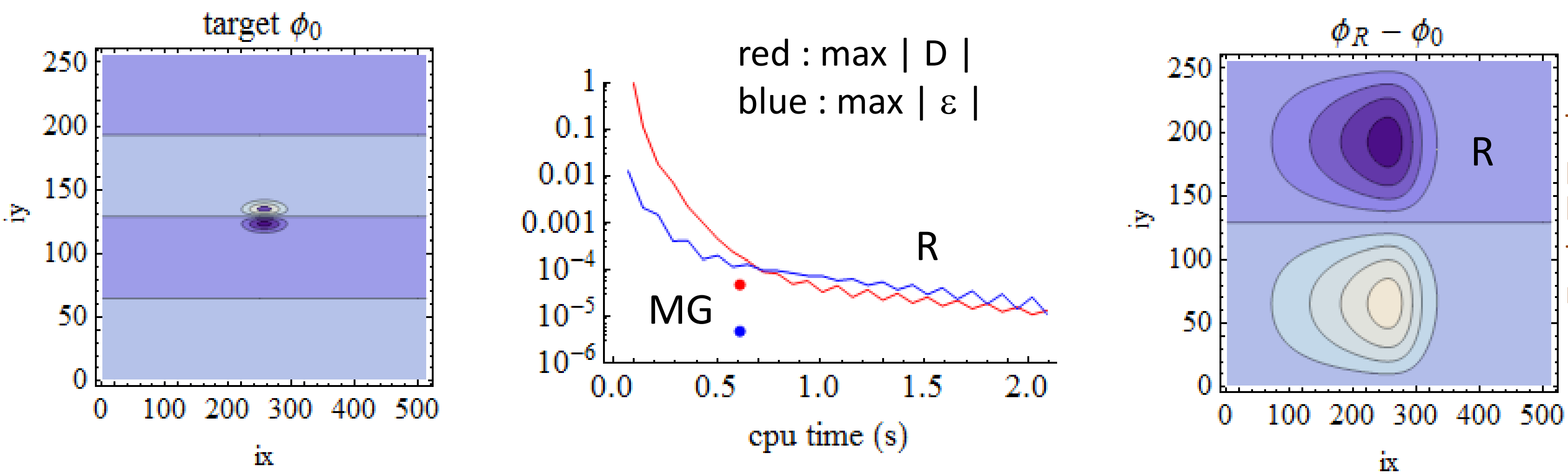
E : exact solve

I : interpolate

How well does it work?

Pure Relaxation (R) on the fine grid vs. the MG V-cycle ($N_C = 3$, $\nu = 2$) Method of Manufactured Solutions (MMS)

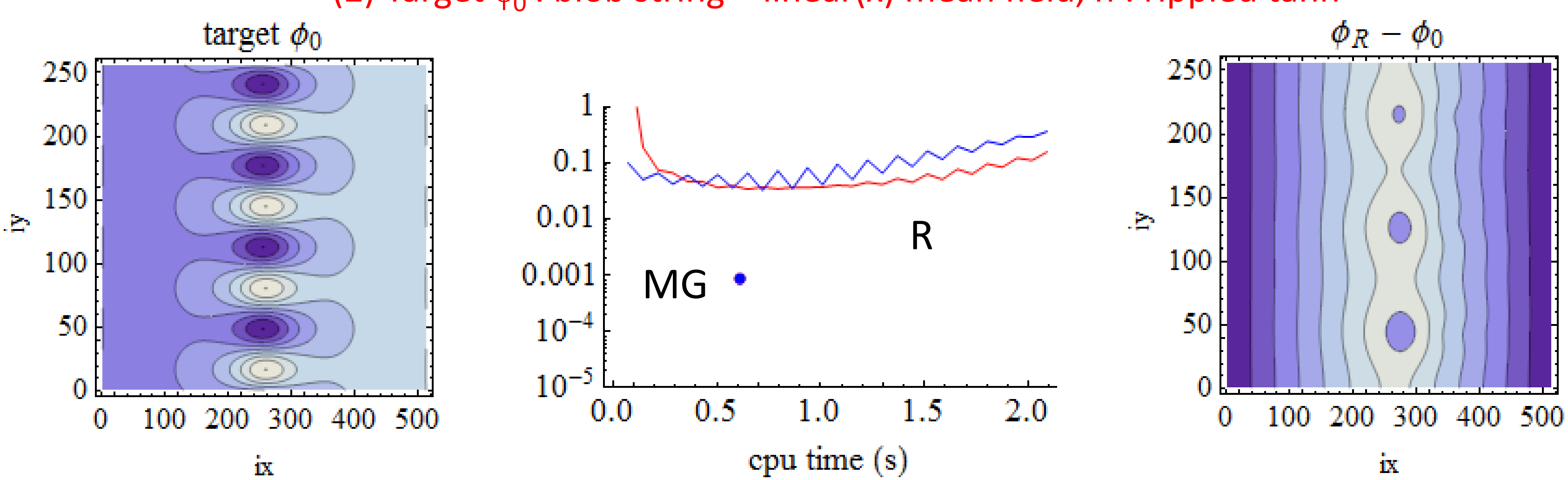
(1) Target ϕ_0 : high-k blob, no mean field, $n = 1.05 - \tanh(x - L_x/2)$



The error persists at long scales for pure relaxation.

Pure relaxation is stable but slow. Relaxation may never achieve MG accuracy on time scales of interest.

(2) Target ϕ_0 : blob string + linear(x) mean field, n : rippled tanh



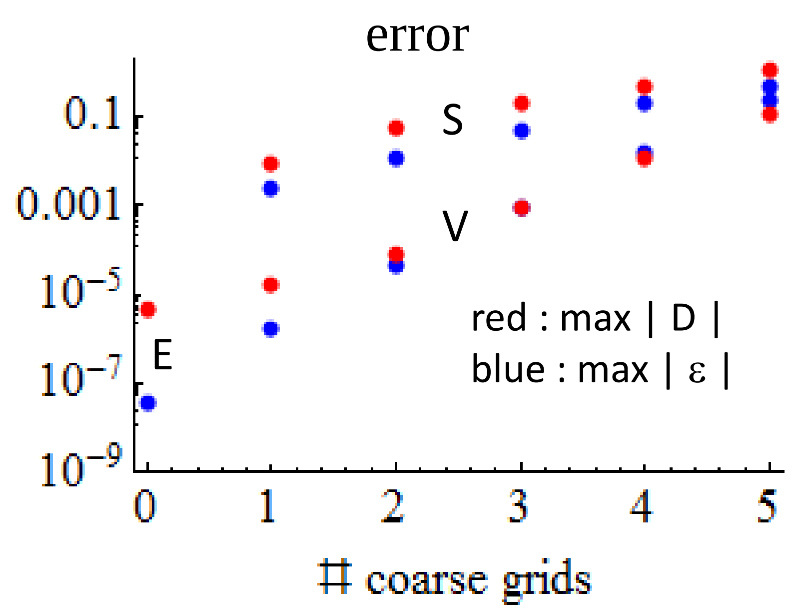
Pure relaxation is unstable at long scales. Relaxation alone <u>cannot</u> achieve MG accuracy.

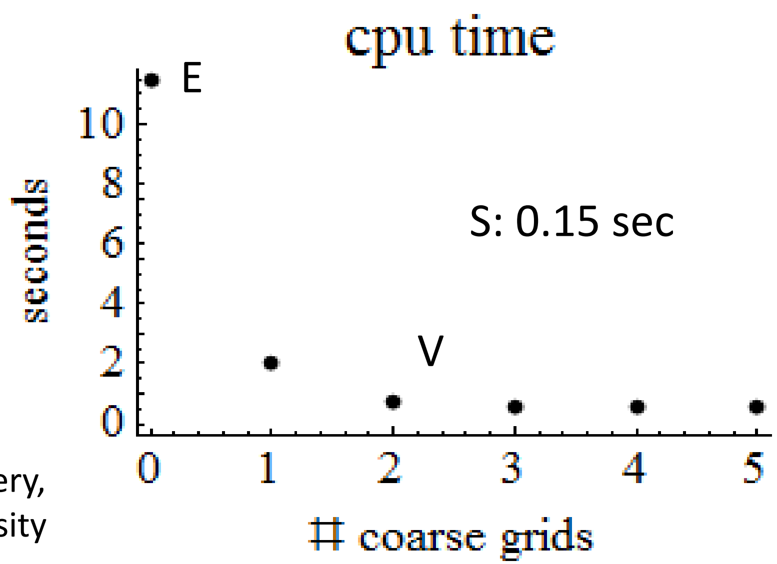
V-cycle accuracy and timing vs. # coarse grids (N_C) (MMS case 2 above)

E :
exact solution on the fine grid
(bi-conjugate gradient method*)

S :
2 relaxations of the initial guess
on the fine grid only

V :
V-cycle
V improves S
(just making sure)





How much error can we tolerate?
Are 2 or 3 coarse grids good enough?
Do MMS results extrapolate to turbulence?

*W. H. Press, S. A. Teukolsky, W. T. Vetterling and B. P. Flannery, Numerical Recipes in Fortran 77, 2nd ed., Cambridge University Press, (1992), pg. 77.

Blob Dynamics
Boussinesq (B) v. non-Boussinesq (E)

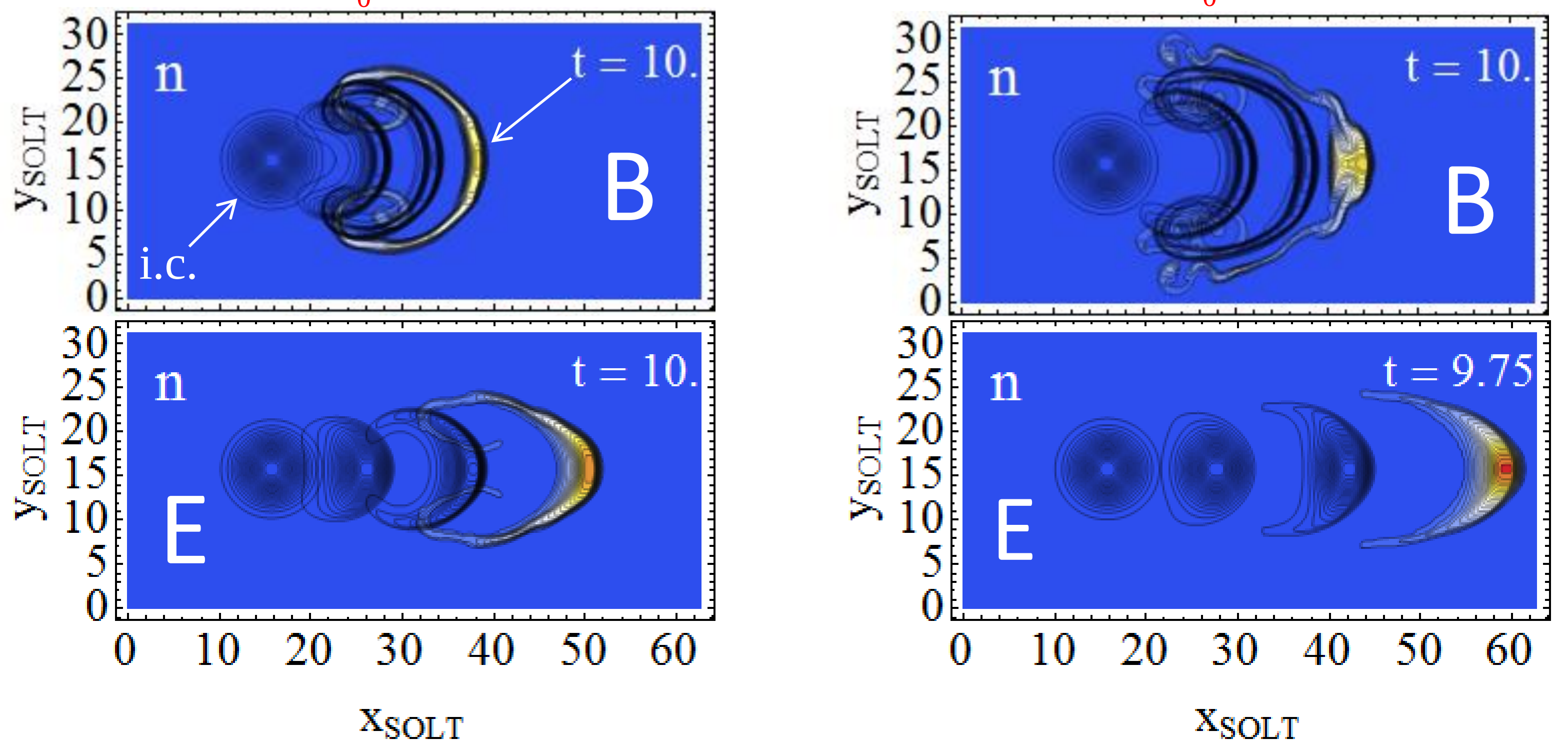
E : Exact solution of the general case

$$\beta = 1, T_e = 1, T_i = 0, \text{no sheath, no drift waves}$$

$$\text{scale invariance } (P_i = 0): t \to \alpha t, \phi \to \phi/\alpha, \beta \to \beta/\alpha^2 (\text{or } \beta \to \beta/\alpha \text{ and } T_e \to T_e/\alpha)$$

$$n(t=0) = n_0 + \text{Gaussian}, \quad \phi(t=0) = 0$$

$n_0 = 0.05$ $\qquad\qquad$ $n_0 = 0.01$



Compared to the general case, Boussinesq blobs are unstable*
and liable to under-estimate turbulent transport in simulations.

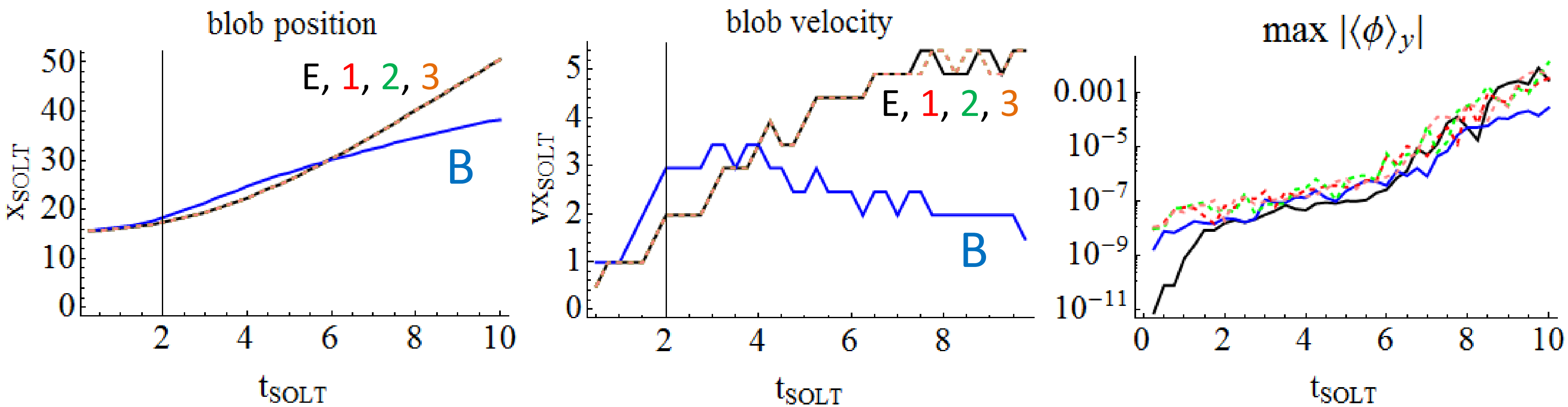
*Related: G.Q. Yu, S.I. Krasheninnikov and P.N. Guzdar, Phys. Plasmas **13**, 042508 (2006).

Blob Dynamics : MG Triumph

Boussinesq (B) v. non-Boussinesq (E) v. MG (N_C = 1,2,3)

E: exact solution of the general case on the simulation grid (256x128),
B: Boussinesq approximation, and MG V-cycle(ν=2), with N_C = 1, 2, 3 coarse grids

$n_0 = 0.05$ case (previous slide)



- Boussinesq is fast but inaccurate.
- All 3 MGs follow the exact solution very well.

cpu sec/step
E : 0.7
1 : 0.23
2 : 0.084
3 : 0.068
B : 0.03

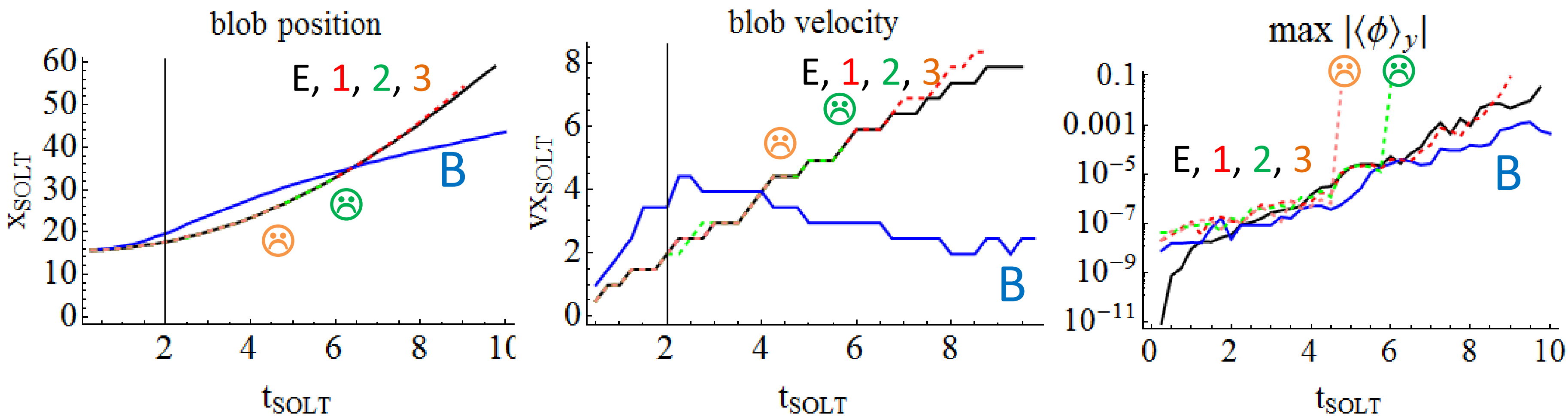
The Ecstasy
MG achieves exact solution accuracy in 1/10 the time, in some test cases.

Blob Dynamics : MG Collapse

Boussinesq (B) v. non-Boussinesq (E) v. MG (N_C = 1,2,3)

E: exact solution of the general case on the simulation grid (256x128),
B: Boussinesq approximation, and MG V-cycle(ν=2), with N_C = 1, 2, 3 coarse grids

$n_0 = 0.01$ case (slide 12)



The 2- and 3-coarse-grid cases bomb due to a numerical instability lurking in the relaxation scheme at long scales: **The instability is apparent in the divergence of the mean field $\langle\varphi\rangle_y$.**

von Neumann multiplier $\sim \dfrac{\nabla n \cdot k}{n k^2}$ $\longrightarrow$

No bombs if

$$\nabla n / n$$

is *small enough*.

The Agony

But how small is that, *à priori*?

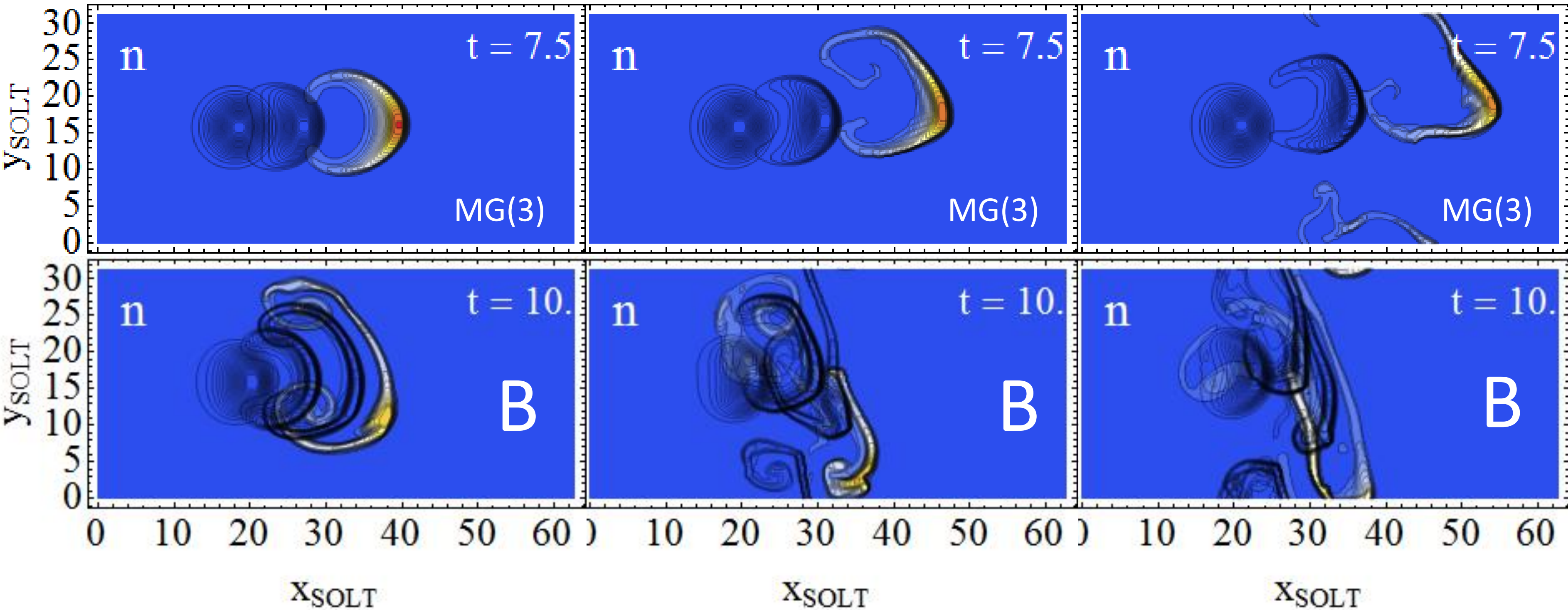
No clue: this is turbulence.

Ion Pressure Effects

Ion pressure enhances curvature drive and blob instability.

$n_0 = 0.05$, Te = 1, β = 1

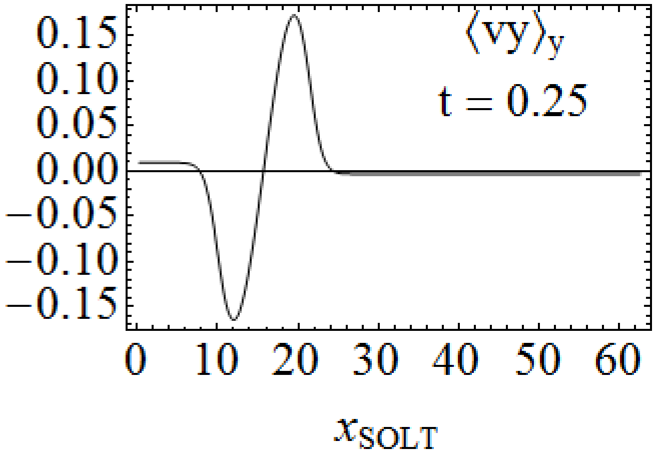
a) Ti = 0.1 b) Ti = 0.5 c) Ti = 1.0



The disintegration of the blob under the Boussinesq approximation may be mistaken for a transport barrier. See "blob trails," next slide.

Ion Pressure Effects
Blob Mean Flow

The blob's ion pressure drives a mean poloidal flow or radial (x) E field.

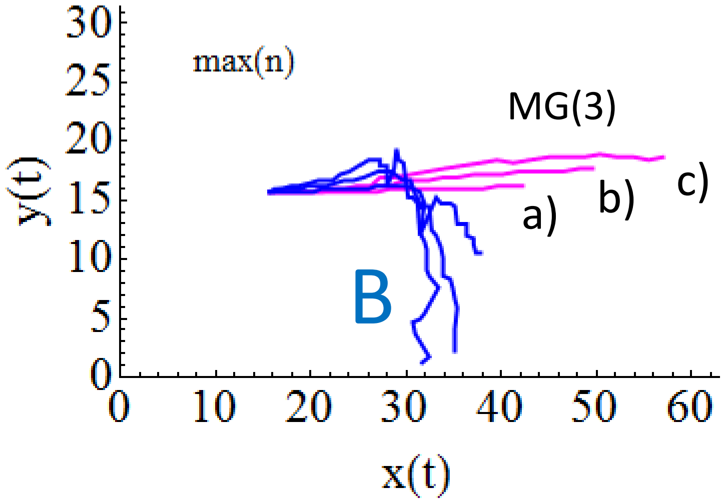


$$\nabla \cdot \left(n \nabla \phi \right) + \rho + \nabla^2 P_i = 0$$

$$\frac{d}{dt} \rho \sim \partial_y P \Rightarrow \langle \rho \rangle_y = 0$$

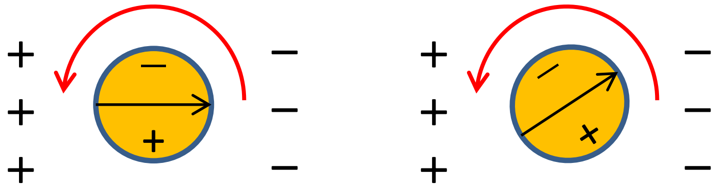
$$E_x = -\partial_x \langle \phi \rangle_y = -\langle vy \rangle_y \sim \partial_x \langle P_i \rangle_y / \langle n \rangle_y$$

blob trails *



In all cases (previous slide) the blobs move upward (y) as they travel outward, initially.

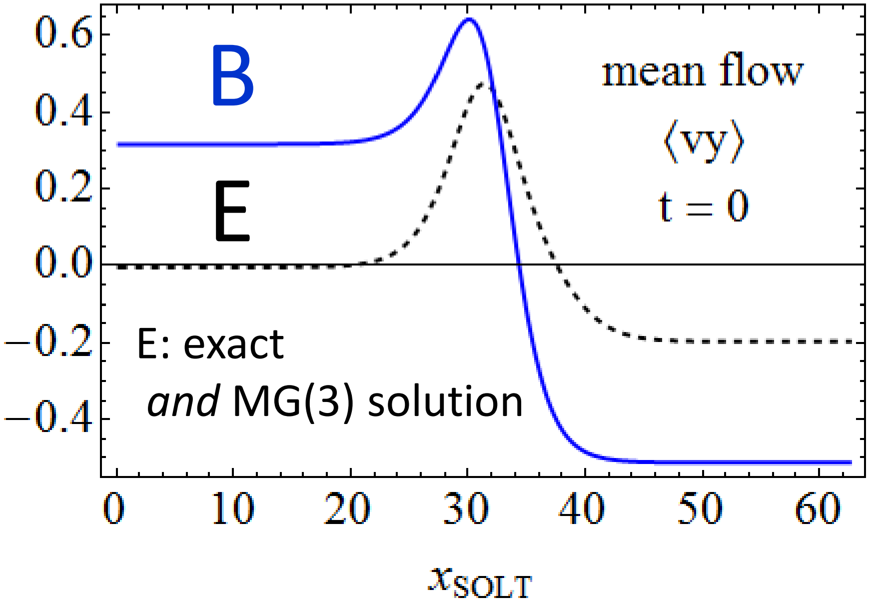
The blob rotates in its mean flow.
i.e.
The radial dipole $\langle \phi \rangle$ exerts a torque on the poloidal (y) dipole $\delta\phi$.

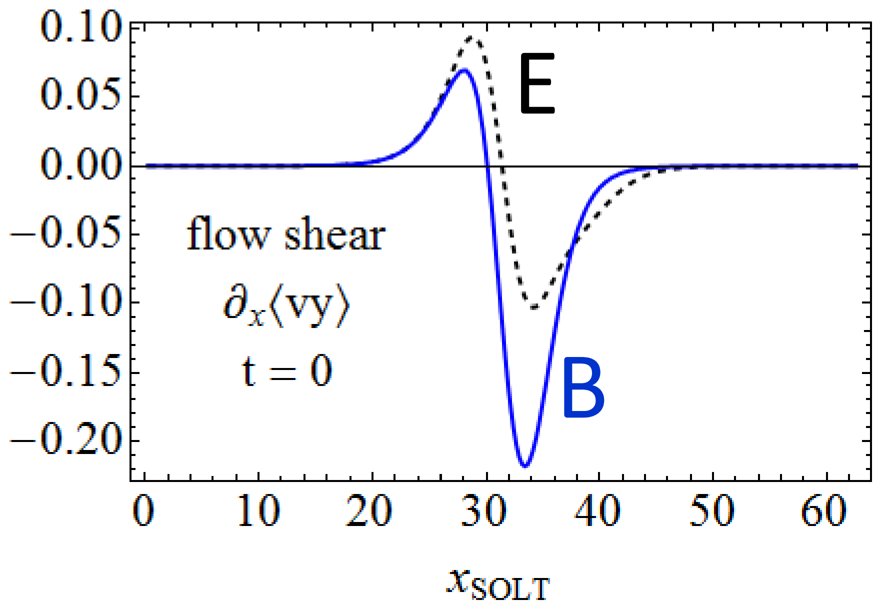


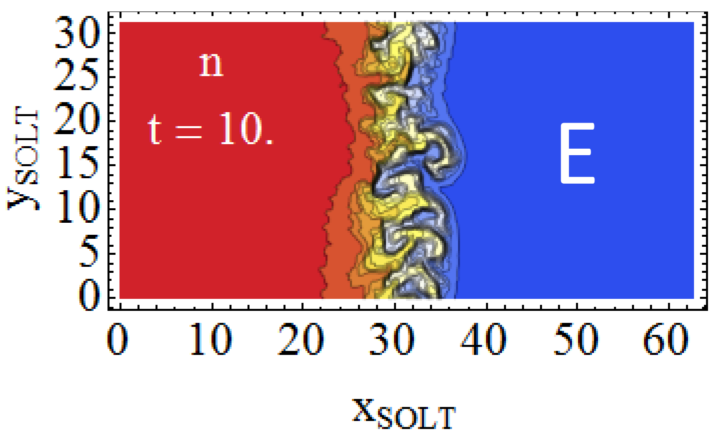
*See J. R. Myra, *Edge Sheared Flows and Blob Dynamics*, Invited Talk YI3.00002, Friday 10:00, this meeting: analysis of blob trails from NSTX and CMOD, and comparison with SOLT simulations.

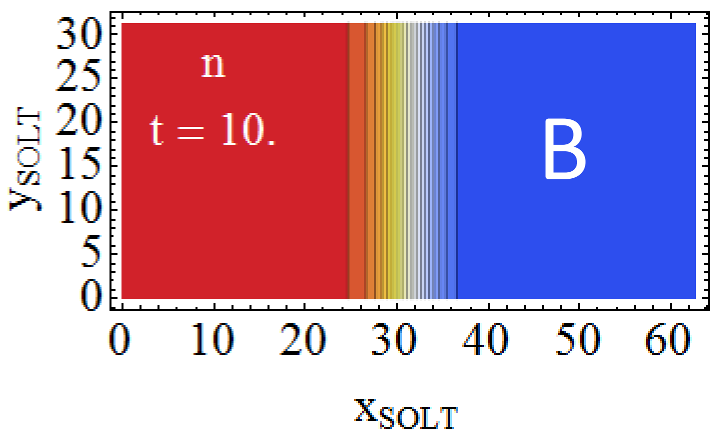
Ion Pressure Effects
Edge Crumble, or Not

Boussinesq (B) over-estimates the ion pressure-driven flow shear and may stifle the instability artificially.









Summary

- We **eliminated the Boussinesq approximation** (B) from the SOLT model and added **ion pressure dynamics**.

- We solved the generalized vorticity equation using a **multigrid (MG) method** and found accuracies of the "exact" solution requiring 1/10 the cpu time.

In comparisons with the exact (or MG) solution we

- **Demonstrated inaccuracies of the B approximation** in
  - **isolated blob dynamics** (blobs are too fragile, turbulent flux may be under-estimated) and
  - **ion pressure profile driven mean flow** (flow shear rate is exaggerated, raising the instability threshold artificially)

❖ **Future work**: L-H transition physics.